***INSTRUCTIONS:***

To receive credit on this exam, you must show all of your work. Grading is based almost entirely on your work: if you have the correct conceptual structure and make a good effort, you will not lose points for arithmetic or algebraic mistakes (except insofar as they might obscure your thinking). Still, read each problem carefully and completely.

You will have an opportunity to submit corrections and discuss the exam in small groups. This work will be incorporated into your grade.

***There will be lots of partial credit. If you are stuck on a problem or want to move on to save time, I still recommend writing out in words what you might do. This can be worth a lot of points!!***

(for instance "I would substitute this polynomial approximation into the rate equation, then expand it and collect terms. Then I would make a comparison to obtain a system of equations to solve for the coefficients of my substitution". This would be nearly full credit!!!!)

***If you need or want a brief break, feel free to step out of the classroom at any time to walk around or go to the bathroom or anything else for a couple minutes. You don't need to ask me, just please refrain from discussing the exam with others during the first portion.***

Name:

Please copy and sign the statement below:

*I agree not to use or access any unauthorized material during this exam.*

Signature:

***Try to relax and do your best. No matter what, you have learned a lot!! Struggling with an exam does not mean you don't understand the material, and we will work together to make sure your course grade reflects your learning and effort.***

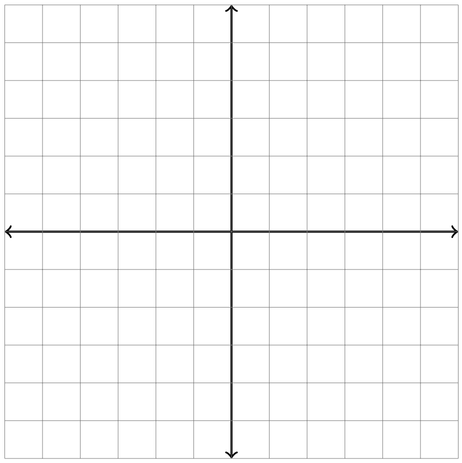
Here are our fundamental functions and their derivatives:

| **Rule** | **Input Function** | **Derivative** |
|---|---|---|
| **Power** | $x^n$ | $nx^{n-1}$ |
| **Exponential** | $e^x, \exp(x)$ | $e^x, \exp(x)$ |
| **Logarithm** | $\ln(x)$ | $\frac{1}{x}$ |

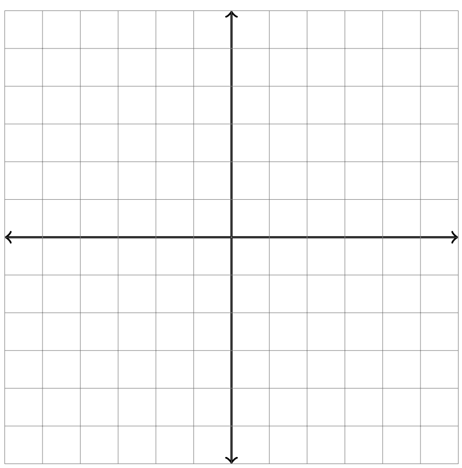
Here are our fundamental rules for calculating derivatives of combinations of functions:

| **Rule** | **Input Function** | **Derivative** |
|---|---|---|
| **Scaling** | $af(x)$ | $af'(x)$ |
| **Sum** | $f(x) + g(x)$ | $f'(x) + g'(x)$ |
| **Product** | $f(x)g(x)$ | $f'(x)g(x) + f(x)g'(x)$ |
| **Quotient** | $\frac{f(x)}{g(x)}$ | $\frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$ |
| **Chain** | $f(g(x))$ | $f'(g(x))g'(x)$ |

**1. In your own words, what does the derivative of $f(x)$ at $x = a$ represent? Include a sketch.**



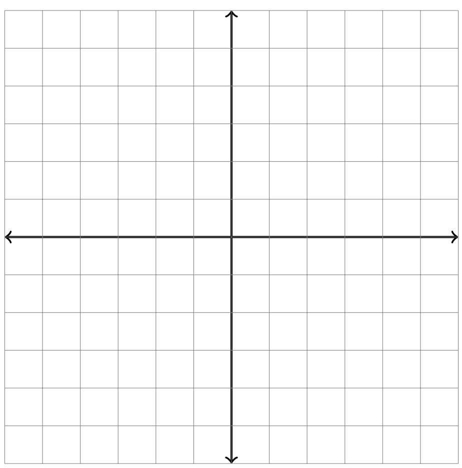
**2. State the fundamental theorem of the derivative. Explain (roughly) why the fundamental theorem of the derivative makes sense. (A picture might help).**



**3. Evaluate any two of the following three limits. You will probably have to factor at least one polynomial.**

$$\lim_{x\to 5} \frac{x^2-25}{x-5} \qquad \lim_{x\to -1} \frac{(2x+2)^2-(2x+2)}{2x+2} \qquad \lim_{x\to 3} \frac{x^2-4x+3}{x-3}$$

**4. Write two limits functions which *CAN NOT* be evaluated at $x = 2$, *BUT* which have a limit as $x$ approaches $2$. One of your functions can be a sketch, but the other should be given by an equation.**



**5. Explain where and how we use limits to give a precise definition of the derivative. A brief 2-3 sentence explanation is sufficient!**

**6. Suppose you introduce** $400$ **swamp rabbits to a lake. There is ample food and this is one of the only rabbit species that can swim, so they adapt readily to this area. As such, they reproduce very rapidly, producing 20 baby rabbits per rabbit per 4 months. Write a rate equation describing this situation, then state an exact solution to the rate equation. Check that your exact solution satisfies the rate equation.**

**7. The rabbits turn out to be trouble makers, damaging local kayaks and canoes. Out of guilt, you offer to replace and rabbit-proof all of the canoes on the lake. This is an expensive undertaking, so you approach a bank for a loan. You are offered a loan of** $\$500,000$ **at a** $5\%$ **interest rate per dollar per year, compounded continuously. Write a rate equation describing the balance on your loan as a function of time (assuming you don't make payments) and an exact solution. Check that your exact solution satisfies the rate equation.**

**8. Explain how the Sum, Product, and Composition rules are each related to operations on lines.**

**9. Choose one:**

**(a) Using the definition of the derivative, justify the product rule. AND use the product rule and the relationship $g(x)g(x)^{-1} = 1$ to write the derivative of $g(x)^{-1}$ in terms of $g(x)$ and $g'(x)$.**

OR

**(b) Using the definition of the derivative, justify the chain rule.**

**10. Show that the rate equation**

$$f'(x) = -x^2 + 3e^x$$

$$f(1) = 4$$

**has a unique solution by using the fundamental theorem of the derivative. Hint: start by taking two solutions $f(x)$ and $g(x)$ to the above. Compare them by subtracting, so you want to show $f(x) - g(x) = 0$. Use the fundamental theorem of the derivative to show that $f(x) - g(x)$ is a constant, then plug in the given initial value to show that that constant is zero.**

**11. Choose three of the following functions and calculate their derivatives.**

$e^x(x^3 - x + 70)$ $\qquad (e^x)^6 - 2(e^x)^2 + 1$ $\qquad \ln(x^4 + 19x^2 + 1)$ $\qquad \ln(5 + 2x)(x^2 - 20)$

***You do not need to simplify your answers. Make sure you clearly indicate which three derivatives you are going to calculate. If you have time to attempt all four, I will grade the best three.***

**12.** **[this is a long question] Determine a degree 2 polynomial approximation for the following rate equation:**

$$f'(x) = \frac{1 + 2x + 3x^2}{f(x)}$$

$$f(0) = 5$$

**Remember the steps. (1) Write** $f(x) = A + Bx + Cx^2 + (error)$**, (2) plug this function into the first part of the rate equation, and expand into polynomials, (3) compare coefficients *only using the constant and linear terms* to obtain some equations involving** $A, B, C$**, (4) use the initial condition to make one more equation, (5) solve the system of equations for** $A, B, C$**, (6) finish by writing** $f(x) = A + Bx + Cx^2 + (error)$ **using your values for** $A, B, C$**.**

**13. Using the approximations $e^x = 1 + x + \frac{1}{2}x^2 + (error)$ and $f(x) = -2 + 2x^2 + (error)$, write a degree two approximation to $g(x) = f(x)e^x$ and a degree two approximation to $h(x) = e^{f(x)}$. From the approximations, determine $g(0), h(0)$ and $g'(0), h'(0)$.**

**14. Explain in your own words why approximations are interesting and/or useful. Why might someone prefer polynomial approximation to discrete approximation?**

**15. Apply the exponential and logarithm rules to the following expressions to make them easier to differentiate. *Do NOT take the derivatives.* Remember $\ln(e) = 1$.**

$\ln(e^{4x+4}(2x^2 - 9))$ $\quad$ $\ln(8xe^{12x}(x^2 + 3x + 9)(x + 2))$ $\quad$ $e^{\ln(8-2)+3x}$ $\quad$ $\ln(2x^4 e^x)$

**16. Choose two expressions from Question 15. Differentiate them. If you didn't finish Question 15, you can differentiate the expressions below as backup. (even if your answers to Question 15 are wrong, you will still get full credit here if you correctly determine the derivative of whatever you wrote above).**

$\ln(21) + \ln(x) + \ln(x^3 - 5)$ $\quad$ $2x^3 + 12x + \ln(5x + 6)$

**17. Let $f(x) = 9x^2 + 2x$. Suppose we have an inverse $g(x)$. Use the relationship $f(g(x)) = x$ to write a rate equation which only involves $g(x)$ and $g'(x)$. Briefly explain how you would use this rate equation to find a degree $4$ approximation to $g(x)$.**

***Do not find this approximation!! It will waste a lot of your time!! Just outline the steps you would follow in enough detail that I can follow your thinking.***

**18. Suppose you are given $f(x) = 4x^2 + 2x + 1$ and a mystery function $g(x)$ such that $g(0) = 3$ and $g'(0) = 2$. Use this information to evaluate the derivatives of $f(x)g(x)$ and $f(g(x))$ at $x = 0$.**

**19. Let $\ell(x) = x - 4$ and $m(x) = 3x$. Given the graph of some function $f(x)$, describe in words how the graph of $m(\ell(f(x))$ is related to the graph of $f(x)$.**

**Challenge Question. Consider the following rate equation that involves *two* derivatives:**

$$f''(x) = f(x)$$

$$f(0) = 1$$

$$f'(0) = 0$$

**Let $g(x) = f'(x)$. Show that $g(x)$ satisfies the rate equation**

$$g''(x) = g(x)$$

$$g(0) = 0$$

$$g'(0) = 1$$

**Use the rate equations and the Fundamental Theorem of the Derivative to verify the following identity. Hint: check the derivative of the left hand side is zero, which means it is a constant. Plug in the initial conditions to check that this constant is $1$.**

$$f(x)^2 - g(x)^2 = 1$$