Previously we learned about some families of interesting functions. Now we will talk about how to combine and transform functions to produce... more functions!

Composition. When sketching a function $f(x)$, we often found it useful to "plug in" some values for x to find a few points on the graph, like $(0, f(0)), (2, f(2)), (-3, f(-3))$. In this situation, we were substituting in numbers for x. But we could do something else (which is not so helpful for sketching) and plug in another function. This is called composition.

If we have two functions $f(x)$ and $g(y)$, then their composition is written $f(g(y))$, and we can write out an expression for it by replacing every x in $f(x)$ with a copy of $g(y)$.

1. $f(x) = x^2 + 1$ and $g(y) = y + 3$:

$$f(g(y)) = (y+3)^2 + 1 = y^2 + 6y + 9 + 1 = y^2 + 6y + 10$$

$$g(f(x)) = (x^2 + 1) + 3 = x^2 + 4$$

2. $f(x) = \sqrt{x+3}$ and $g(y) = y^2$:

$$f(g(y)) = \sqrt{y^2 + 3}$$

$$g(f(x)) = (\sqrt{x+3})^2 = x + 3$$

From these examples, you can see that what you get depends on the order you compose in.

Usually, compositions give rise to much much more complicated functions than what you started with, and in particular to completely different looking graphs. However, if you compose with a *linear map* then the complexity stays roughly the same, and it's straightforward to describe the graph of the composition in terms of the original two.

Transformations. There are two fundamental (linear) transformations: translation and scaling. Together, these can be combined into any linear transformation. These transformations shift the graph around and squish or stretch it.

Shifts. A graph can be shifted horizontally or vertically or both. These transformations correspond to composition with a linear map from the right or left, respectively.

Let $\ell(t) = t + a$. If we compose from the left with a function $f(x)$, then we get $\ell(f(x)) = f(x) + a$. We can see that this takes every point on $f(x)$ and moves it vertically by a (upward if a is positive, downward if negative).

On the other hand, we can compose with $\ell(t)$ on the right, to get $f(\ell(t)) = f(t + a)$. It's a little harder to read off what this transformation does, but we can plug in a couple values for t to see what happens:

$$t = 0 \quad \Rightarrow f(a)$$

$$t = 1 \quad \Rightarrow f(1 + a)$$

$$t = -1 \quad \Rightarrow f(-1 + a)$$

See the pattern? It moves the graph of $f(x)$ so that $f(a)$ is over zero. This means that when a is positive, the graph of $f(x)$ moves ***left*** (right if a is negative).

Scaling. Instead of shifting, we can scale a graph vertically or horizontally.

Let $\ell(t) = at$. Composing with $f(x)$ from the left, we get $\ell(f(x)) = af(x)$. We can see that this takes every point on $f(x)$ and multiplies it by a, so the graph is stretched vertically by a factor of a when a is positive. If a is negative, then the graph flips, while stretching by the same factor.

As before, we can put $\ell(t)$ on the right, to get $f(\ell(t)) = f(at)$. Let's plug in some values to see what happens here:

$$t = 0 \quad \Rightarrow f(0)$$

$$t = 1 \quad \Rightarrow f(a)$$

$$t = -1 \quad \Rightarrow f(-a)$$

$$t = 2 \quad \Rightarrow f(2a)$$

$$t = -2 \quad \Rightarrow f(-2a)$$

The pattern here is a little harder to see, but what happens is that we have sort of "changed units" on the input to make 1 unit correspond to a of the original units. If we take $a = 5$, then this means that the new function puts $f(5)$ over 1 and $f(2 * 5) = f(10)$ over 2, so it *pulls in* far away parts of $f(x)$ towards zero.

This results in a horizontal scaling with a factor of $\frac{1}{a}$, when a is positive. As before, if a is negative we will still scale by a factor corresponding to $|a|$, but multiplying by a negative number flips the two sides of the input axis, and so we flip the graph over the vertical axis in addition to scaling.