Last time we talked informally about power series. But what may have concerned you then, and definitely on the homework thinking questions, is that it's not always possible to plug numbers into a power series, unlike a regular polynomial.

The way around this is to view these infinite polynomials as limits of finite polynomials:

$$1 + x + x^2 + x^3 + ...$$

is approximated by 1, then $1 + x$, then $1 + x + x^2$, and so on. So we can evaluate this series at x if that sequence of numbers **converges** to a single finite number.

For example, let's look at what we get from the above power series at $x = \frac{1}{2}$:

$$1, 1 + \frac{1}{2}, 1 + \frac{1}{2} + \frac{1}{4}, 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}, ...$$

Which is the same as

$$1, 1.5, 1.75, 1.875, ...$$

And if we keep going, this list of numbers heads right to 2. So the power series converges at $\frac{1}{2}$ to a value of 2.

Here's another example:

$$3, 3.1, 3.14, 3.1415, ...$$

which converges to π, if we fill out the rest of the digits the right way.

Unfortunately, it can be hard to tell that a series or sequence is actually converging. The following is called the harmonic series:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + ...$$

and even though the approximations grow very slowly, and the amount added at each step decreases to zero, it doesn't converge. If you go out far enough, your approximation can be made arbitrarily large. This is called **divergence**. Other series, like $1+1+1+...$ are less sneaky about diverging.

Checking that power series converge is a special case of checking that a **limit** exists. In the case of power series, the limit we are looking at is the limit of the sequence of finite approximations. But we can take limits of other things too.

For instance, if a function is not defined at some input, we could try to study the behavior near that point and take a limit towards that input to get information.

The limit of $f(x)$ as x approaches a real number a is denoted

$$\lim_{x \to a} f(x)$$

For very nice functions, like polynomials, $\lim_{x\to a} f(x) = f(a)$. Intuitively, this means that $f(a)$ is determined by knowing all the values $f(x)$ takes close to a. Such functions are called **continuous** because they don't have any jumps or gaps.

In this class, you won't have to worry about checking for convergence and divergence (that's a MATH0100 topic) and you will always be told. We only need to keep it in mind, because the fact that power series don't have to be defined at every possible input means that remembering their domain (the places where they converge) is important.

Likewise for limits, we will not take a very careful approach – just an intuitive understanding of "getting close to" will suffice for this class. From this, we can identify several limit rules:

1. **Additivity**. Suppose the limits of both f and g exist as $x \to a$. Then the limit of $f(x)+g(x)$ as $x \to a$ is defined and

$$\lim_{x\to a} f(x) + \lim_{x\to b} g(x) = \lim_{x\to a} [f(x) + g(x)]$$

2. **Multiplicativity**. Suppose the limits of both f and g exist as $x \to a$. Then the limit of $f(x)g(x)$ as $x \to a$ is defined and

$$\left[\lim_{x\to a} f(x)\right] \left[\lim_{x\to b} g(x)\right] = \lim_{x\to a} [f(x)g(x)]$$

3. **Composition**. Suppose the limit of $f(x)$ as $x \to a$ exists and is equal to b. Suppose that g is continuous and defined at b. Then the limit of $f(g(x))$ as $x \to a$ exists and

$$\lim_{x\to a} g(f(x)) = g(\lim_{x\to a} f(x))$$

Every function we've looked at closely so far is continuous, and so we don't get anything interesting out of limits - we can just plug in the value and evaluate the function. So now we will learn about our first and most fundamental kind of possibly-discontinuous function: division. Since you can't divide by zero, any time division appears there is a discontinuity.

For example,

$$f(x) = \frac{1}{x^2}$$

This function is not defined at zero, and if we try to look at $\lim_{x\to 0} \frac{1}{x}$, there is a problem right away: from either side the values of $f(x)$ grow arbitrarily large (to $-\infty$ on the left and $+\infty$ on the right).

On the other hand, division itself does not guarantee poor behavior; look at

$$f(x) = \frac{2x^2 + 3x^3 + 4x^4}{x^2}$$

Then $f(x)$ can't be defined at zero because we aren't able to divide by it, but

$$\lim_{x\to 0} \frac{2x^2 + 3x^3 + 4x^4}{x^2} = 2$$

and if we fill in the point $(0, 2)$ on the graph of that function, it makes it continuous! So sometimes limits allow us to "patch holes"