Now that we have had some practice computing derivatives by hand, directly from the definition, we will learn a few rules to simplify these calculations.

It is important to note that what we did before remains valuable: *previous calculations determined not just the derivative, but also exact error terms.* Sometimes it's useful to keep at least some track of the error term. The tricks below will allow us to calculate a derivative, but we won't know much about what the error looks like.

We will state three rules (power, sum, product, and quotient) and verify the last three (sum, product and quotient). You should know and be comfortable with the explanations of the sum, product, and quotient rules, but you do not need to remember how we arrived at the power rule (which was done loosely in class).

(1) Power Rule: this says that if $f(x) = cx^n$, then $f'(a) = nca^{n-1}$, where c is any coefficient.

(2) Sum Rule: this is the nicest of our rules, which says that the derivative of $f(x) + g(x)$ at $x = a$ is $f'(a) + g'(a)$, so the derivative "plays well" with sums. To see why this is true, we just work right from the definitions:

$$f(x) = f(a) + f'(a)(x - a) + error, \quad g(x) = g(a) + g'(a)(x - a) + error$$

and then add the two:

$$f(x) + g(x) = f(a)g(a) + (f'(a) + g'(a))(x - a) + error$$

This matches the shape of our definition of the derivative, and so we see that $f'(a) + g'(a)$ is the derivative of $f(x) + g(x)$ at a.

Notice that the Sum Rule says we can differentiate a polynomial by differentiating each of its terms separately and adding them; those terms can be differentiated dirrectly with the power rule. Together these make it straightforward to find the derivative of a polynomial that doesn't require expanding huge products. *However, we won't obtain information on the error term, as mentioned previously. This is usually not a concern, but worth keeping in mind.*

(3) Product Rule: since the derivative works well with addition, we might wonder next what it does with multiplication. The statement is more complicated, but the verification is not. The derivative of $f(x)g(x)$ at $x = a$ is

$$f'(a)g(a) + f(a)g'(a)$$

We will check this the same way we checked the Sum Rule: by just multiplying $f(x)$ and $g(x)$ when expressed in terms of the derivative. Remember that the error is so small that anything times the error is still error.

$$\begin{aligned} f(x)g(x) &= (f(a) + f'(a)(x - a) + error)(g(x) = g(a) + g'(a)(x - a) + error) \\ &= f(a)g(a) + (f'(a)g(a) + g'(a)f(a))(x - a) + f'(a)g'(a)(x - a)^2 + error \\ &= f(a)g(a) + (f'(a)g(a) + g'(a)f(a))(x - a) + error \end{aligned}$$

Compared to the definition of the derivative of $f(x)g(x)$, we see that $f'(a)g(a) + g'(a)f(a)$ is the correct derivative!! Notice that to get to the third step, we included $f'(a)g'(a)(x-a)^2$ with the error. This makes sense: it's a high power of $x - a$ so it must be very small near $x = a$.

(4) Quotient Rule: after multiplication, division is our next question. This is our messiest rule so far. The derivative of $f(x)/g(x)$ at $x = a$ is

$$\frac{g(a)f'(a) - f(a)g'(a)}{g(a)^2}$$

Now, notice that $f(x)/g(x) = f(x)g(x)^{-1}$, so if we know the derivative of $g(x)^{-1}$ then we can use product rule to get the final answer.

To find that, we just start the way we always do: right from the definitions. The defining property of $g(x)^{-1}$ is that $g(x)g(x)^{-1} = 1$. Let's take the derivative of both sides at $x = a$.

$$0 = g'(a)g(a)^{-1} + [g(x)^{-1}]'(a)g(a)$$

Rearranging isolate the term we want:

$$[g(x)^{-1}]'(a) = -\frac{g'(a)}{g(a)^2}$$

Now we can solve our original question!! By the product rule, the derivative of $f(x)/g(x)$ is

$$f'(a)/g(a) + f(a)[g(a)^{-1}]'(a) = \frac{f'(a)}{g(a)} - f(a)\frac{g'(a)}{g(a)^2} = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}$$

The last and most important topic is the **fundamental theorem of the derivative**. In spite of the name, it's actually quite straightfoward, and follows naturally from our work on approximation. This deceptively useful fact is that:

$$\text{if } f'(a) = 0 \text{ for all } a \text{ then } f(x) \text{ is a constant}$$

This is highly reasonable: we know that the derivative measures the local linear growth of a function. If this growth is zero everywhere, then the function is best approximated by a horizontal line at every point, which could only happen for a function which is itself constant.

The main way we will use this fact is to check identities relating functions - if we can't directly work out the original claim, the next best thing is taking the derivative, which generally simplifies functions (e.g. for polynomials it lowers the degree).