Today we defined the exponential function, and worked out some examples. Common situations where exponential functions show up are: population growth, interest accumulation, chemical reactions.

Consider our example from class. Pandas are very slow breeders, and might only produce two baby pandas per hundred pandas per year. Therefore, the population **grows at a rate** of $\frac{2}{100}P(t)$ where P is the number of pandas (t in years). As we know, the derivative exactly measures the rate of growth at any given time, and so this means that

$$P'(t) = \frac{2}{100}P(t)$$

with some initial panda population $P(0)$. This gives us a **rate equation** that we could use to find $P(t)$, which would give us a formula for the number of pandas present at time t.

Any time we have a rate equation of the form

$$f'(t) = rf(t)$$

we expect to get a solution which is an exponential function.

For simplicity, we will first study functions where $r = 1$ and $f(0) = 1$. It turns out that we can build all the other exponential functions from this one.

What we saw last time from our sketching, is that if we know a point on a function and the derivative, we can essentially reconstruct the function (by making extremely accurate approximations). This is how we will define exponential functions.

The **exponential function** is the unique function determined by the following **rate equation**:

$$f'(x) = f(x)$$

$$f(0) = 1$$

(this is called a rate equation because the derivative tells us the "rate of growth" of a function, and we have an equation describing it). From the rate equation, we can sketch approximations to this function. For instance, at zero we have the approximation

$$f(x) = f(0) + f'(0)x + \cancel{error} = 1 + x$$

because $f'(0) = f(0) = 1$ by definition. Using this, we could determine for example that $f(0.5) = 1+0.5 = 1.5$ (approximately). Then from the definition we would also know that $f'(0.5) = f(0.5) = 1.5$. We can fit these together into another approximation:

$$f(x) = f(0.5) + f'(0.5)x + \cancel{error} = 1.5 + 1.5x$$

Proceeding in this fashion, we get a decent approximation, and we see from it that this function very rapidly increases.

We name this function $f(x) = e^x$ or sometimes $\exp(x)$. This function will form the basis of our investigation of growth. However, we need one more ingredient to account for the r in the general exponential rate equation.

Another derivative rule. We will need the following useful property: if $g(x) = f(ax)$ for some constant a, then $g'(x) = af'(ax)$. In class we checked this geometrically for e^{4x}, and the same argument works in general.

The idea is that $g(x)$ is the same as $f(x)$ but squished inward by a factor of a. This means at point x, the slope of the approximating line through x is a times larger than it was at the corresponding point of $f(ax)$, and so $g'(x) = af'(ax)$.

Applying the above rule to $f(x) = \exp(rx)$, we see that

$$f'(t) = rf(t)$$

All that is left to do is incorporate the initial condition $f(0)$. But this is straightforward: multiplication *on the outside* by a constant simply multiplies the slope of the approximating line by the same amount, and $\exp(0) = 1$, so if we multiply $\exp(rx)$ by the starting initial value, we will get the desired function.

In other words, starting from the rate equation

$$f'(t) = rf(t)$$

$$f(0) = C$$

the exact solution is $f(x) = C\exp(rx) = Ce^{rx}$. Let's double-check ourselves!!

First,

$$f(0) = C\exp(r0) = C\exp(0) \underset{\exp(0)=1}{=} C \times 1 = C$$

so this function has the right starting value. Secondly,

$$f'(x) = C[\text{derivative of } \exp(rx)] = Cr\exp(rx) = rf(x)$$

and so the function has exactly the right growth.

If we return to our panda example, and take a starting population of 1000 pandas, our rate equation is

$$P'(t) = \frac{2}{100}P(t)$$

$$P(0) = 1000$$

and so the panda function is $P(t) = 1000e^{\frac{2}{100}t}$. Using a calculator, this would tell us the number of pandas present at a given time.

Next time, we will use calculus to verify some properties of exponential functions that you are probably familiar with, like $e^{a+b} = e^a e^b$, as well as some more examples.