Up until now, we have defined interesting functions (the exponential ones) by means of a rate equation. But this doesn't necessary tell us how to compute their values!!

We already have one method of approximation, which we will now refer to as discrete approximation. This works quite well, but it has to be redone every time we want to compute a new approximation. Today we will introduce polynomial approximations, which in principle we only have to determine once, and then we can plug anything we want in. The reason we want to have a polynomial is that these are functions we can really plug numbers into and evaluate (in a sense they are the only ones where we can do this!).

Calculators and computers can't actually evaluate e^x at most numbers: they use a very good and the numbers they print off for you, like $e^1 = 7.38...$ are simply very good approximations.

The starting point is to pretend we can write a function as a power series or polynomial:

$$f(x) = A + Bx + Cx^2 + Dx^3 + ...(error)$$

This approximation should be good near zero. We've actually be using such approximations for this entire course!! The derivative is defined based on the best *linear* approximation. All we are doing now is adding in extra coefficients to improve the precision of our estimates.

Any time you come across a mysterious rate equation "in nature" you can try to find polynomial approximations to study its behavior. Sometimes there will be no solutions at all in terms of functions we are familiar with, and then approximations like this are the best you can do.

Just as degree 1 approximations (the derivative!) have been very important to us, we will see (after the midterm) that degree 2 approximations are also quite important and carry valuable information. They're the next easiest to compute after the linear approximation.

What we worked out in class is an approximation for e^x (see worked out worksheet and recitation solutions for more functions, $\sqrt{x+1}$ and $x^{1/3}$). Remember that e^x is defined by the rate equation

$$f'(x) = f(x)$$

$$f(0) = 1$$

We start by writing an approximation

$$f(x) = A + Bx + Cx^2 + Dx^3 + Ex^4 + (error),$$

where the coefficients A, B, C, D, E unknown numbers that we want to solve for. If we can find them, then we will have a degree 4 approximation. We claim that this should satisfy the rate equation (as always, neglecting error). In other words, the derivative of the approximation,

$$f'(x) = B + 2Cx + 3Dx^2 + 4Ex^3 + (error)$$

should equal the original approximation after we toss out the error. (we are assuming that the error has a small derivative too!)

If we put $f(x)$ and $f'(x)$ into the rate equation, we get an equality of polynomials:

$$B + 2Cx + 3Dx^2 + 4Ex^3 + (error) = A + Bx + Cx^2 + Dx^3 + Ex^4 + (error)$$

Since the left hand side is only accurate up to degree 3, we will drop Ex^4 from the right hand side. Then, we can compare the coefficients to get a series of equalities:

$$B = A$$

$$2C = B$$

$$3D = C$$

$$4E = D$$

Finally, the given value $f(0) = 1$ gives us

$$A = 1$$

and we have a system of 5 equations in 5 unknowns. Solving it gives us

$$A = 1 \quad B = 1 \quad C = \frac{1}{2} \quad D = \frac{1}{6} \quad E = \frac{1}{24}$$

which we can now use to write out our approximation:

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + (error).$$

Plugging in small values for x into just the polynomial should give us reasonable estimates for the values of e^x. If we want greater accuracy, we could repeat the process above with more terms.