1. Combine the following functions using the exponential rules. Then, differentiate these (simpler) expressions (using composition rule).

(a) $e^x e^{2x} e^{3x} e^{4x}$

$$= e^{x+2x+3x+4x} = e^{10x} = g(f(x))$$

$$\begin{cases} f(x) = 10x \\ g(x) = e^x \end{cases}$$

comp rule

$$g(f(x)) \longrightarrow g'(f(x))\,f'(x) = e^{10x}\,10$$

(b) $e^{x^2} e^{2x} e^1$

$$= e^{x^2+2x+1} = f(g(x))$$

$$\left.\begin{matrix} f(x)=e^x & f'(x)=e^x \\ g(x)=x^2+2x+1 & g'(x)=2x+2 \end{matrix}\right]\text{comp rule}\quad f'(g(x))g'(x) = e^{x^2+2x+1}(2x+2)$$

(c) $e^x e^{-x} e^2$

$$e^{x-x+2} = e^2$$

constant

derivative zero

(d) $e^{x-2x^2+x^3} e^{3x^2}$

$$= e^{x^3+\underbrace{3x^2-2x^2}+x}$$

$$\begin{cases} f(x) = e^x & f'(x)=e^x \\ g(x) = x^3+x^2+x & g'(x) = 3x^2+2x+1 \end{cases}$$

$$= e^{x^3+x^2+x} = f(g(x)) \xrightarrow{\text{comp rule}} f'(g(x))g'(x) = e^{x^3+x^2+x}(3x^2+2x+1)$$

2. Calculate a degree 4 approximation of e^x. Use this to estimate e^2. Does this seem like a reasonable estimate (e^2 is roughly 7.5)? What could you do to improve the estimate?

$$e^x \begin{cases} f'(x) = f(x) \\ f(0) = 1 \end{cases}$$

Set $f(x) = A + Bx + Cx^2 + Dx^3 + Ex^4 + \cancel{F}$

$f'(x) = B + 2Cx + 3Dx^2 + 4Ex^3 + \cancel{5}$

Compare: $A = B$, $B = 2C$, $C = 3D$, $D = 4E$

Also: $f(0) = 1$ means $A = 1$

Solve: $A = B = 1$, $C = \frac{1}{2}$, $D = \frac{1}{6}$ $E = \frac{1}{24}$

Approximation: e^x near $1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$

$e^2 \approx 6$ ~~sort~~ of close. Higher degree for more accuracy.

3. Using your previous degree 4 approximation of e^x, find a degree 8 approximation of e^{x^2} (hint: plug in x^2!).

done!

$$e^{x^2} \approx 1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \frac{1}{24}x^8 + \cancel{E}$$

4. Suppose we have a mystery function $f(x)$ which is approximated by

$$2 + x + x^2 + 2x^3 + x^4 + \textbf{error}.$$

What is the derivative of $f(x)$? Use the above to approximate the derivative of $f(x)e^x$ at zero.

$$f'(x) = 1 + 2x + 6x^2 + 4x^3 + \cancel{E}$$

$$f(x)e^x \xrightarrow[\text{at } 0]{\text{product rule}} f'(x)e^x + f(x)e^x$$

$$\hookrightarrow \underset{1}{f'(0)}\,\underset{1}{e^0} + \underset{2}{f(0)}\,\underset{1}{e^0} = 1 + 2 = 3$$

5. Suppose we have two functions $f(x)$ and $g(x)$ such that

$$f(g(x)) = x$$

Differentiate the above expression. If $f(x) = x^2$ and $g(x) = \sqrt{x}$, what does this tell us about the derivative of $\sqrt{x}$?

$$f(g(x)) = x \xrightarrow{\text{differentiate both sides}} f'(g(x))\,g'(x) = 1$$

If $f(x) = x^2$, $g(x) = \sqrt{x}$ then $f(g(x)) = x$

$f'(x) = 2x$

$$\hookrightarrow 2g(x)\,g'(x) = 1$$

6. Using your answer to Question 5, write down a rate equation satisfied by $\sqrt{x+1}$? Using that rate equation, find a degree 2 polynomial approximation to $\sqrt{x+1}$. (this is a very difficult problem!!)

$$\begin{cases} 2g(x)g'(x) = 1 \\ g(0) = 1 \end{cases}$$

$$\sqrt{x+1} \approx 1 + \frac{1}{2}x + \frac{1}{4}x^2$$

$$g(x) = A + Bx + Cx^2 +$$

$$g'(x) = B + 2Cx +$$

$$1 = 2g(x)g'(x) = 2(A + Bx + Cx^2 + \;)\cdot(B + 2Cx + \;)$$

$$1 = 2AB + (2B^2 + 2AC)x +$$

Compare:

$$\left.\begin{aligned} 1 &= 2AB \\ 0 &= 2B^2 + 2AC \\ 1 &= g(0) = A \end{aligned}\right\} \quad \begin{aligned} A &= 1 \\ B &= \frac{1}{2} \\ C &= -\frac{1}{4} \end{aligned}$$