Let $f(x) = e^{2x}(x^2 + 4x + 1)$. We will find a degree 2 polynomial approximation to this in two ways.

1. Recall that e^x can be approximated as $1 + x + \frac{1}{2}x^2 + error$. Use this to write a degree two approximation to e^{2x} (i.e. substitute $2x$ into the known approximation).

$$e^{2x} = 1 + (2x) + \frac{1}{2}(2x)^2 + \text{error}$$
$$= 1 + 2x + 2x^2 + \text{error}$$

2. Since our function is e^{2x} times $x^2 + 4x + 1$, we can make an approximation by multiplying our approximation from Question 1 with the polynomial $x^2 + 4x + 1$. Do this below, expanding the product completely and then collecting the terms into a single polynomial.

$$(1 + 2x + 2x^2)(x^2 + 4x + 1)$$
$$= 1 + 6x + 11x^2 + 10x^3 + 2x^4$$

3. Because we only took degree 2 precision in writing e^{2x}'s approximation, the degree 3 and higher terms of the previous equation are part of the error. Take your answer to Question 2 and truncate to just a degree 2 approximation. This is our final answer for the approximation.

$$f(x) = 1 + 6x + 11x^2 +$$

4. Another method we can use is the derivative method from the previous homework. Write $f(x) = A + Bx + Cx^2 + (error)$. By evaluating $f(x)$ at zero, we get $f(0) = A$. So what is A?

$$A = f(0) = \underbrace{e^{2\cdot 0}}_{1}\underbrace{(0^2 + 4\cdot 0 + 1)}_{1} = 1 \checkmark$$

5. Next, take the derivative of $f'(x)$ and our approximation $A + Bx + Cx^2 + (error)$. You will need the product rule to take the derivative of $f'(x)$. Your answer should look like $f'(x) = (messy\ function) = B + 2Cx + (error)$.

$$B + 2Cx + \ldots = f'(x) = 2e^{2x}(x^2 + 4x + 1) + e^{2x}(2x + 4)$$

6. If we take our answer from the previous step and evaluate at $x = 0$, the right hand side is B. Evaluate the left hand side (the messy function) to determine what B equals.

$$B = \underbrace{2e^{2\cdot 0}}_{2} \times \underbrace{(0^2 + 4\cdot 0 + 1)}_{1} + \underbrace{e^{2\cdot 0}}_{1}\underbrace{(2\cdot 0 + 4)}_{4}$$
$$= 2 + 4 = 6$$

$2e^{2x}(x^2+4x+1) + e^{2x}(2x+4)$

7. Take the derivative of the messy function above. This is the second derivative of $f(x)$, which we denote $f''(x)$ (two $'$s for taking two derivatives). This will need even more product rule. You will end up with

$2C$

$$(extremely\ messy\ function) = 2C + (error)$$

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By evaluating the extremely messy function at zero, you can determine C. What is it?

$4e^{2x}(x^2+4x+1) + 2e^{2x}(2x+4) + 2e^{2x}(2x+4) + e^{2x}(2)$

@ $x=0 \rightsquigarrow 4\cdot 1 + 2\cdot 4 + 2\cdot 4 + 1\cdot 2 = 22 \rightsquigarrow C=11$

8.Fit together your values for A, B, C determined in Questions 4-7 to write an approximation $f(x) = A + Bx + Cx^2 + (error)$.

$f(x) = 1 + 6x + 11x^2 + \cancel{E}$

9. Compare your two approximations! (From Questions 3 and 8). They should be equal. Which method do you prefer?

Yes!! The first method :)

10. Use the approximation you calculated to determine $f(0)$ as well as $f'(0)$ *without* using the product rule.

$f(0) = 1 + 6\cdot 0 + 11\cdot 0^2 + \cancel{E} = 1$

$f'(0) = 6 + \cancel{11\cdot 2\cdot 0} + \cancel{E}$

$= 6$

Composition/Product Practice. You do not need to do all of these, but do at least four or five. If you find them difficult, do more for practice (or if you find them easy and quick, you might as well do more anyway!!) For each of the following functions, determine whether it is a product of composition, then determine functions $f(x)$ and $g(x)$ so that the given function is either the product $f(x)g(x)$ or the composition $f(g(x))$, and then take the derivative using the appropriate rule. Your answer should be a word (product or composition), two functions ($f(x)$ and $g(x)$), and another function (the derivative).

prod $(3x^9 + 2x + 1)(x^2 + x)$	comp e^{5x+3x^2+x}	Comp $(e^x)^8 - 9e^x + 5 + (e^x)^3$	comp $(e^{2x})^2 + (e^{2x}) + 1$
comp $\ln(x^3 + 2)$	comp $\ln(e^x + 3)$	x comp $\ln(x)^2 - 4\ln(x) + 2$	$e^{2\ln(x)+4}$ → expand. no rule! or comp
$e^{(}5x)\ln(x^2-x-20)$ prod & comp	$\ln(1+x+x^4)(x^3-2x+1)$ prod & comp	$\ln(2x)^3+\ln(2x)-15$ comp in comp	$\ln(x^4+4x+1)$ comp