Let $f(x) = 2x^2 + x$. Suppose we have a mystery function $g(x)$ such that $f(g(x)) = x$. As always: we can try to understand $g(x)$ by calculus, which means we need to find $g'(x)$, then a rate equation, then an approximation to $g(x)$ from the rate equation.

1. Differentiate the expression $f(g(x)) = x$. On the left hand side use the composition rule. Your answer should look like $f'(\textit{something}) * g'(x) = 1$.

$$f'(g(x))\,g'(x) = 1$$

composition rule

2. Fill in $f'(x)$ above to obtain an expression relating $g(x)$ and $g'(x)$. [some kind of equality]

$$f'(x) = 4x+1 \longrightarrow (4g(x)+1)\,g'(x) = 1$$

3. In the previous step you should have obtained ($\cancel{2}$ 4 $g(x)+1)g'(x) = 1$. Let's take $g(x)$ to be such that $g(0) = 0$. Write these together as a complete rate equation:

$$\begin{cases} (4g(x)+1)\,g'(x) = 1 \\ g(0) = 0 \end{cases}$$

4. Find a degree 2 polynomial approximation of $g(x)$ using the rate equation above. Remember we start by writing $g(x) = A + Bx + Cx^2 + (\textit{error})$, and so $g'(x) = B + 2Cx$. We plug both of these values into the rate equation, then expand and collect terms. This will give us a polynomial whose coefficients involve A, B, C on the left hand side.

Comparing coefficients on the left and right, we obtain various equations relating A, B, C. Include the initial value $A = g(0) = 0$ and solve this system of equations for A, B, and C. For your final answer, write out $g(x) = A + Bx + Cx^2 + (\textit{error})$ using the values of A, B, C that you found.

$$1 = (4(A + Bx + Cx^2) + 1)(B + 2Cx)$$

$$= 4AB + B + 4B^2x + 8ACx + 2Cx$$

$$\rightarrow 4AB + B = 1, \quad 4B^2 + 8AC + 2C = 0$$

$A = 0$

$B = 1$

$C = -2$

$g(x) = x - 2x^2 + \text{error}$

5. Take just your polynomial approximation $A + Bx + Cx^2$ from the previous step. Plug it in to $f(x)$ and expand (write out $f(A+Bx+Cx^2)$). You will get some polynomial. Does it appear interesting?

$$f(x-2x^2) = 2(x-2x^2)^2 + (x - 2x^2)$$

$$= x + \underline{\quad}_{\text{no } x^2!} - \underbrace{8x^3 + 8x^4}_{\text{error}}$$

Another way to calculate polynomial approximations by using *multiple derivatives.* This can be a handy trick.

Let's try to find a degree 3 polynomial approximation to $f(x) = e^x$, and so we start by writing $f(x) = A + Bx + Cx^2 + Dx^3 + (error)$.

1. Plug zero into $f(x)$ and its polynomial approximation above. This will determine the value of A. [a number]

$f(0) = A + B(0) + C(0) + D(0) = A$

$= e^0 = 1$

2. Take the derivative of both sides of $f(x) = A + Bx + Cx^2 + Dx^3 + (error)$. You will end up with

$$f'(x) = B + 2Cx + 3Dx^2 + (error)$$

Remember $f(x) = e^x$ and so $f'(x) =$??? Then plug zero into both sides above. This will determine the value of B. [a number]

e^x

$B = f'(0) = e^0 = 1$

3. Now take *another* derivative of our equality from the previous step. You should have had $f(x) = f'(x) = B + 2Cx + 3Dx^2 + (error)$ and so its derivative is

$$f''(x) = 2C + 6Dx + (error)$$

Once again, what is $f''(x)$? Plug zero into both sides and solve for C. [a number]

e^x

$2C = f''(0) = e^0 = 1 \rightsquigarrow C = \frac{1}{2}$

4. Yet again start with the equation above, $f(x) = f'(x) = 2C + 6Dx + (error)$ and differentiate it. Plug zero into both sides and solve for D. [a number]

$6D = f'''(0) = e^0 = 1 \rightsquigarrow D = \frac{1}{6}$

5. Looking at your previous answers, you now have values for A, B, C, D. Use these to write down the degree 3 approximation to e^x. [a polynomial] If you look back at your notes, we get the same polynomial this way as we did with our other method! Which do you prefer?

$f(x) = \underbrace{1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3}_{\text{matches work in class}} + (error)$