**Please let me know if you find any mistakes!!**

# Contents

## Notes



## Worksheets



## Homework



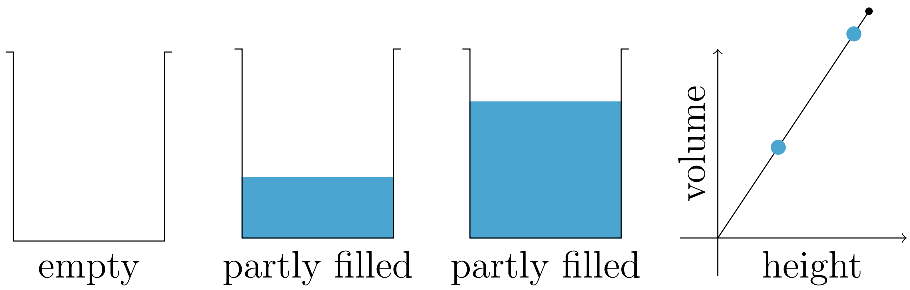
## Recitation

## Worksheet Solutions

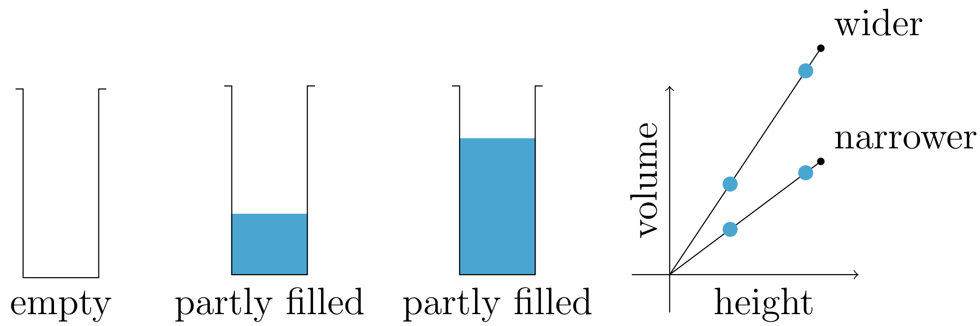
For the syllabus, please see our course website:
https://mark-sing.com/math50fall22/

Today we introduced **calibration functions** for bottles/beakers. These graphs tell you the total volume in a container when it is filled to a certain height. In other words, they are a **function** from heights to volumes.

For example, consider a simple beaker, filled up to two different heights. We can plot the three different heights and volumes on a graph, and it looks like the calibration function will be a line. The dot "capping" the line is a reminder that the calibration function is only defined up until the bottle is completely full:




Now, suppose that we repeated this with a narrower bottle, half the width, but using the same heights. The smaller bottle will have a smaller volume at the same heights, so we end up with a shallower line, half as steep:




What about a bottle whose width isn't constant? Width corresponds to steepness, this means that the the steepness of the calibration function isn't constant – it is a curve. For lots of examples, see the worksheet and its solutions. Given some bottle, the width at a given height tells you how quickly the calibration function is growing near that height. This is an instance of the *first definition* of calculus from the syllabus: it is the the study of functions in terms of their growth. Later, we will revisit calibration functions in detail to make the connection precise.

A function is a rule that takes in an input (like a number) and produces some output (usually another number). You could think of an animal as a function that takes food as input, and outputs the behavior of that animal!! (e.g. a rabbit eats hay and turns it into a heartbeat, hopping, waste, etc).

For instance, $f(x) = x^2 + 2x + 1$ is a **quadratic** function, which is a special kind of **polynomial** (see below). For this function $f(x)$, the input is a number $x$, and the output is the value of $x^2 + 2x + 1$. For instance, $f(2) = 2^2 + 2 * 2 + 1 = 9$.

It is important to note that functions don't all have to be called $f(x)$. The names "$f$" and "$x$" can be changed. For instance, when talking about **V**elocity as a function of **T**ime, we might write $v(t)$ so that the letters remind us of the quantities they represent.

Most commonly in sciences, functions describe a relationship between two interesting numerical quantities. In physics, you might be interested in speed as a function of time. In chemistry, you might be interested in the concentration of a molecule over time.

Some functions we will encounter in this course are: linear, quadratic, polynomials, square root, and piecewise functions. Linear and quadratic functions are actually special kinds of polynomials, but they're also of independent interest. Later on, we will learn about exponential, logarithmic, and trigonometric functions (you don't need to know these now!).

Here are descriptions of some important functions. For pictures, see the worksheet solutions. ***For the purposes of calculus, linear functions are the most important, followed closely by quadratic functions. Make sure you understand them!!***

**Linear:** A linear function is one that can be written in the form $f(x) = mx+b$, for constants $m$ and $b$. The value $m$ is called the **slope**. For instance, $f(x) = 2x + 1$ or $f(x) = -3x + 5$ are both linear functions corresponding to $m = 2, b = 1$ and $m = -3, b = 5$, respectively. When graphed, these functions look like a line.

**Quadratic:** A quadratic function is one that can be written in the form $f(x) = ax^2 + bx + c$, for constants $a, b, c$, where we require that $a$ is not zero (if $a = 0$, then we end up with a linear function!). For instance, $f(x) = x^2$ and $f(x) = x^2 - 2x - 3$ are both quadratic functions. When graphed, they make an upward- or downward-facing "U" shape. They open *upward* when $a$ is *positive*, and *downard* when $a$ is *negative*.

**Polynomial:** Both linear and quadratic functions are special kinds of polynomials. A polynomial is, roughly, any function that can be formed only by using addition and multiplication. The following are all polynomials:

$$f(x) = x^3 - 3x + 1 \qquad g(x) = x + x - 3x + 5xx^2 + 1$$

$$h(x) = x^{55} - x^3 + 1 \qquad j(x) = (x^2 + 3x + 1)(x^4 - 3)$$

Every polynomial can be "expanded" into an expression only involving whole number powers of $x$ (meaning things like $x^3, x, x^8$, but not $x^{-1}$ or $x^{2.1}$ or $x^{1/2}$) multiplied by some numbers and added together. In this form, we say that the "degree" is the highest power of $x$ that appears in the polynomial. For instance, quadratics all have degree 2.

In the examples above, the degrees are, in order, $1, 3, 3, 55, 6$.

**Square root:** This is our first function that is not a polynomial. The square root of $x$, denoted $\sqrt{x}$, is defined to be the number that squares to $x$. Since the square of a number is always positive, this means that the square root *is only defined for positive inputs.*

**Absolute value:** This function is also not a polynomial. The absolute value of $x$ is written $|x|$. It takes in the number $x$ and makes it positive. For instance $|5| = 5$ doesn't change $x$ because it is already positive, while $|-2| = 2$, changing from negative to positive two. Combinations of linear and absolute value functions will look like a bunch of lines connected at sharp corners.

**Piecewise functions:** These are our "strangest" functions. A piecewise function is any function that looks like a combination of other functions tossed together. The only examples we know so far are for calibration functions.

Remember that for oddly shaped bottles – such as the "ink bottle" which went from straight to sloped sides, the graph has two different regions, one corresponding to the calibration of a straight-side bottle, and another to the calibration function of a slanted-side bottle. These different "pieces" are why such functions are called piecewise.

Another piecewise function is the absolute value: for negative inputs $x$, the absolute value returns the line $y = -x$, while for positive inputs $x$, it just gives $y = x$. So it is a piecewise function formed of two lines.

Previously we learned about some families of interesting functions. Now we will talk about how to combine and transform functions to produce... more functions!

**Composition.** When sketching a function $f(x)$, we often found it useful to "plug in" some values for $x$ to find a few points on the graph, like $(0, f(0)), (2, f(2)), (-3, f(-3))$. In this situation, we were substituting in numbers for $x$. But we could do something else (which is not so helpful for sketching) and plug in another function. This is called composition.

If we have two functions $f(x)$ and $g(y)$, then their composition is written $f(g(y))$, and we can write out an expression for it by replacing every $x$ in $f(x)$ with a copy of $g(y)$.

1. $f(x) = x^2 + 1$ and $g(y) = y + 3$:

$$f(g(y)) = (y+3)^2 + 1 = y^2 + 6y + 9 + 1 = y^2 + 6y + 10$$

$$g(f(x)) = (x^2 + 1) + 3 = x^2 + 4$$

2. $f(x) = \sqrt{x+3}$ and $g(y) = y^2$:

$$f(g(y)) = \sqrt{y^2 + 3}$$

$$g(f(x)) = (\sqrt{x+3})^2 = x + 3$$

Sketch these examples or make a table to see that what you get from the composition depends on the order: $f(g(x))$ does not usually equal $g(f(x))$.

Usually, compositions give rise to much much more complicated functions than what you started with, and in particular to completely different looking graphs. However, if you compose with a *linear map* then the complexity stays roughly the same, and it's straightforward to describe the graph of the composition in terms of the original two.

**Transformations.** There are two fundamental (linear) transformations: translation and scaling. Together, these can be combined into any linear transformation. These transformations shift the graph around and squish or stretch it.

**Translation.** A graph can be shifted horizontally or vertically or both. These transformations correspond to composition with a linear map from the right or left, respectively. Let's work this out in a little more detail:

Let $\ell(t) = t + a$. If we compose from the left with a function $f(x)$, then we get $\ell(f(x)) = f(x) + a$. We can see that this takes every point on $f(x)$ and moves it vertically by $a$ (upward if $a$ is positive, downward if negative).

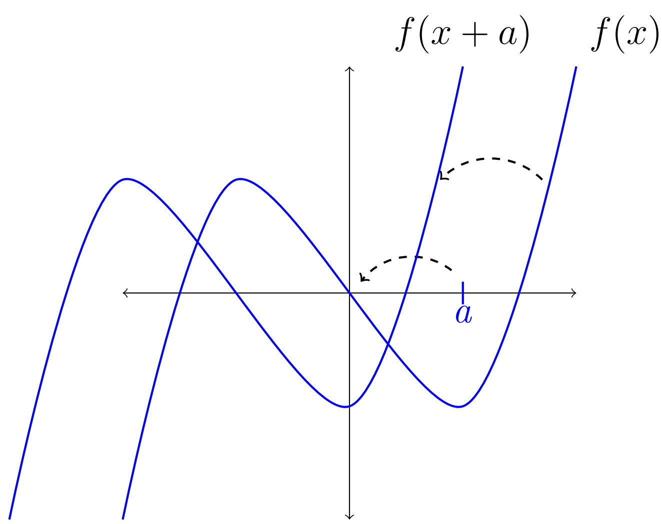
On the other hand, we can compose with $\ell(t)$ on the right, to get $f(\ell(t)) = f(t + a)$. It's a little harder to read off what this transformation does, but we can plug in a couple values for $t$ to see what happens:

$$t = 0 \quad \Rightarrow f(a)$$

$$t = 1 \quad \Rightarrow f(1 + a)$$

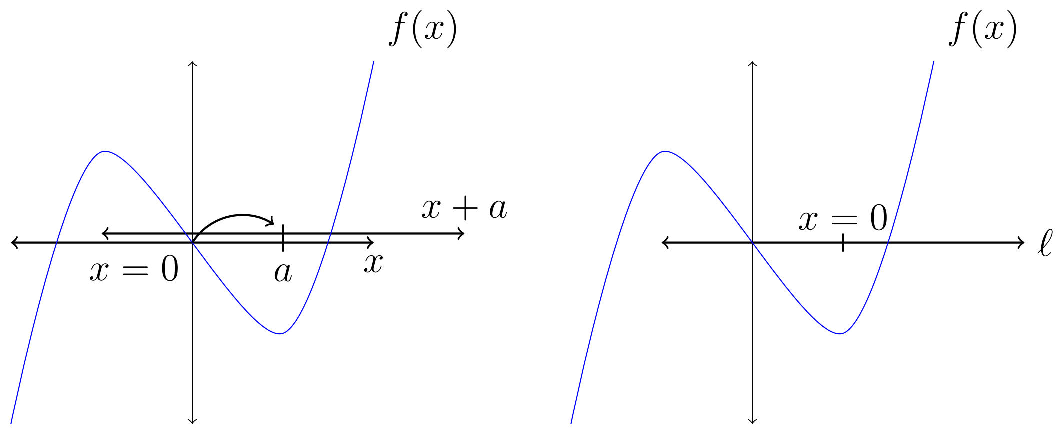
$$t = -1 \quad \Rightarrow f(-1 + a)$$

See the pattern? It moves the graph of $f(x)$ so that $f(a)$ is over zero. This means that when $a$ is positive, the graph of $f(x)$ moves ***left***; right if $a$ is negative. One way to picture this operation is that we are grouping the point $x = a$ and the graph of $f(x)$ together as a single object, and sliding them together to place $a$ over zero:



There is another way to think about $f(x+a)$ which makes it more similar to the behavior of $f(x)+a$. For $f(x)+a$, it's pretty clear that adding $a$ moves the graph of $f(x)$ up by $a$, since we are adding the $a$ to $f(x)$.

For $f(x+a)$ we are adding $a$ to $x$, so *it should "move $x$ by $a$"*: we can achieve the same effect by moving the $x$-axis by $a$ units to the right. When $a$ is positive, the axis moves *right* (a more intuitive direction) which has the visual effect of making the graph of $f(x)$ move *left*!




**Scaling.** Instead of shifting, we can scale a graph vertically or horizontally.

Let $\ell(t) = at$. Composing with $f(x)$ from the left, we get $\ell(f(x)) = af(x)$. We can see that this takes every point on $f(x)$ and multiplies it by $a$, so the graph is stretched vertically by a factor of $a$ when $a$ is positive. If $a$ is negative, then the graph flips, while stretching by the same factor.

As before, we can put $\ell(t)$ on the right, to get $f(\ell(t)) = f(at)$. Let's plug in some values to see what happens here:

$$t = 0 \quad \Rightarrow f(0)$$

$$t = 1 \quad \Rightarrow f(a)$$

$$t = -1 \quad \Rightarrow f(-a)$$

$$t = 2 \quad \Rightarrow f(2a)$$

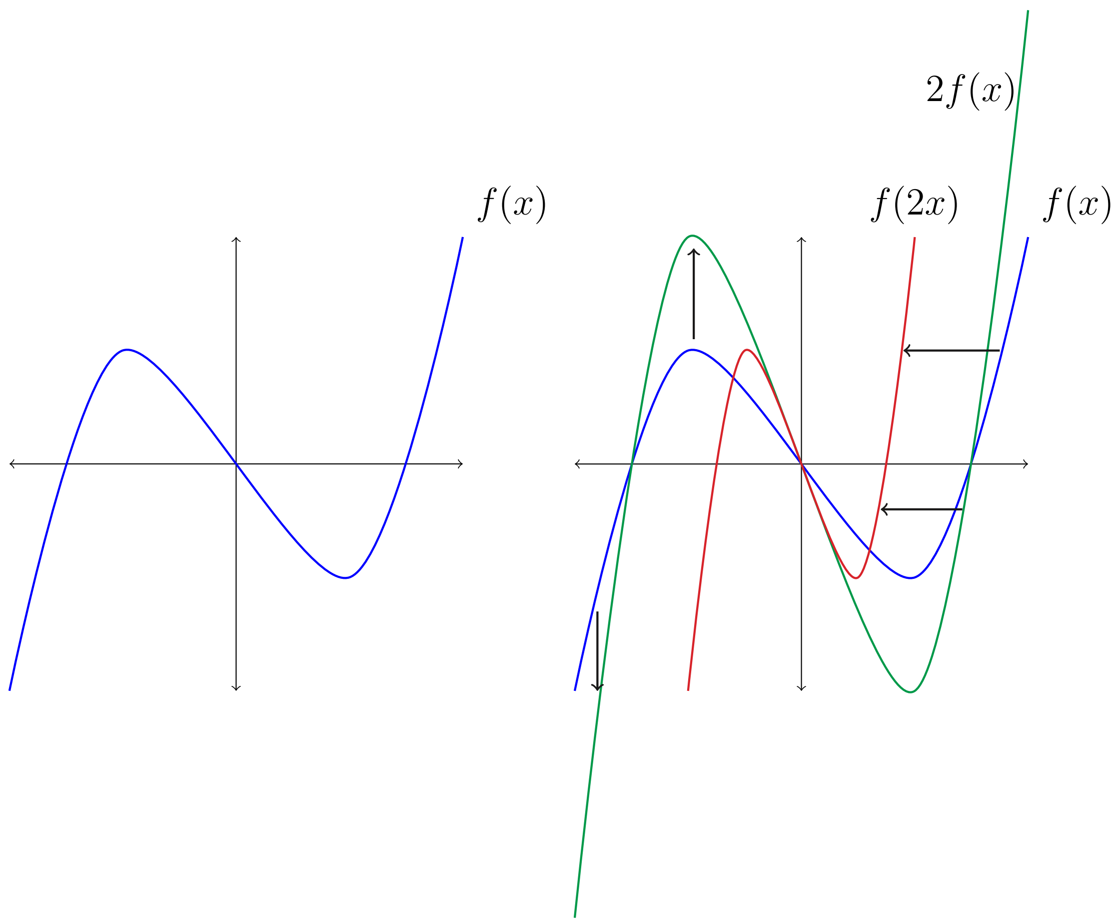
$$t = -2 \quad \Rightarrow f(-2a)$$

The pattern here is a little harder to see, but what happens is that we have sort of "changed units" on the input to make 1 unit correspond to $a$ of the original units. If we take $a = 5$, then this means that the new function puts $f(5)$ over 1 and $f(2*5) = f(10)$ over 2, so it *pulls in* far away parts of $f(x)$ towards zero.

This results in a horizontal scaling with a factor of $\frac{1}{a}$, when $a$ is positive. As before, if $a$ is negative we will still scale by a factor corresponding to $|a|$, but multiplying by a negative number flips the two sides of the input axis, and so we flip the graph over the vertical axis in addition to scaling.

Exactly as we saw with translations, we can interpret the outer application $\ell f(x)$ as acting to scale $f$ by $a$, and the inner application $f(\ell(x))$ as a *scaling of the $x$-axis by $a$*. This stretches out the $x$-axis while holding the graph of $f(x)$ fixes, and hence smaller $x$-values are pulled out to lie under values of $f$ from larger $x$-values.

Let's look at some examples where $\ell(x) = 2x$. Then $\ell(f(x)) = 2f(x)$ is a vertical stretch, depicted in green, while $f(\ell(x))$ is an inward "smush" by a factor of 2, depicted in red.



**Addition and Multiplication.** Besides composition, there are two other ways that we can combine functions: addition and multiplication. These are much more straightforward than composition:

Let $f(x) = x^2 + 2x$ and $g(x) = x - \sqrt{x}$. Then it's not hard to check that:

$$\begin{aligned} f(x) + g(x) &= (x^2 + 2x) + (x - \sqrt{x}) &= x^2 + 3x - \sqrt{x} \\ f(x)g(x) &= (x^2 + 2x)(x - \sqrt{x}) &= x^3 - x^2\sqrt{x} + 2x^2 - 2x\sqrt{x} \end{aligned}$$

(carefully distribution the multiplication in the second row yourself)

When sketching, you can form a table for $f(x) + g(x)$ by taking your tables for $f(x)$ and $g(x)$ alone and adding together the functions' entries (not the input entries!). Likewise for multiplication:

$$f(x) = x^2 + 2 \qquad g(x) = x - 1$$

| $x$ | $f(x)$ | $g(x)$ | $f(x)g(x)$ |
|---|---|---|---|
| $-2$ | $6$ | $-3$ | $-18 = 6 \times (-3)$ |
| $-1$ | $3$ | $-2$ | $-6 = 3 \times (-2)$ |
| $0$ | $2$ | $-1$ | $-2 = 2 \times -1$ |
| $1$ | $3$ | $0$ | $0 = 3 \times 0$ |
| $2$ | $6$ | $1$ | $6 = 6 \times 1$ |
| $3$ | $11$ | $2$ | $22 = 11 \times 2$ |

Alternatively, we can directly expand the product

$$f(x)g(x) = (x^2 + 2)(x - 1) = x^3 - x^2 + 2x - 2$$

and form a table directly, skipping the intermediate steps. However, it is usually easier to keep the two intermediate steps, since it breaks the calculation into more reasonably-sized pieces.

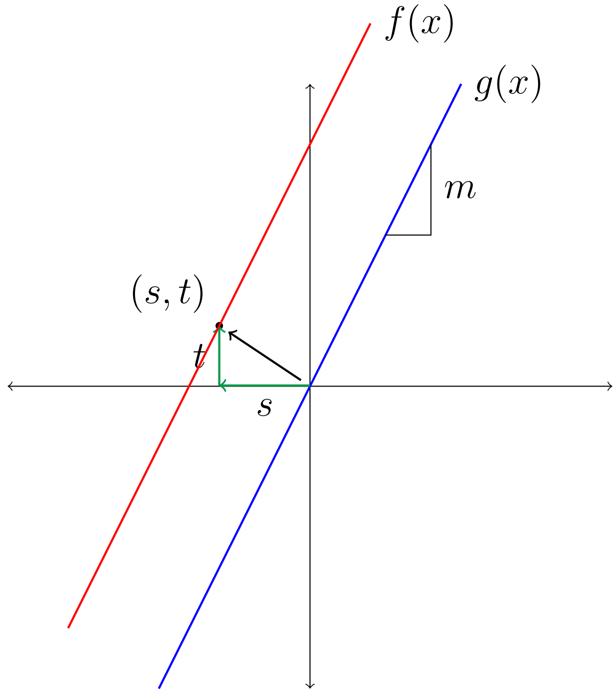
**Polynomials.** In light of the above, we see that polynomials stand out as a basic class of functions: the composition, translation, scaling, sum, and product of two polynomials is always a number of polynomial. This makes them the smallest group of functions that you can build out of these simple operations, and in some ways makes them easier to work with than other functions like $\sqrt{x}$.

One of the goals of our calculus course is to approximate difficult functions like $\sqrt{x}$ (and trig functions, logarithms, and exponentials) by polynomials. With a good enough approximation, we can learn a lot about these functions in spite of their complexity.

The most important functions in calculus are linear functions (followed closely by quadratic functions and higher degree polynomials). At its core, most of calculus revolves around approximating functions with lines. For this to be useful to us, we first need to understand lines a little bit better!

**Point-slope form.** Often, lines are described by the following information: the slope of the line, and a point the line passes through.

Suppose we are given a slope $m$ and a point $(s, t)$ and want to describe the line through this point. To do so, we can start with a line $g(x)$ through $(0, 0)$ with slope $m$, and then apply the translations we learned about last time to make it pass through $(s, t)$:



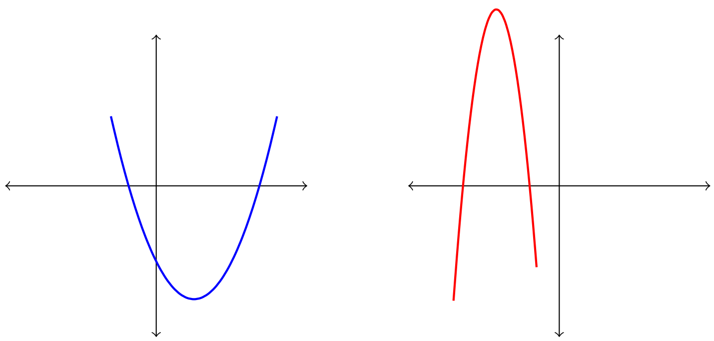

So we want to translate *horizontally by* $s$ and then *vertically by* $t$. We can achieve the horizontal shift with composition on the right by $\ell_h(x) = x - s$, and the vertical shift by composing on the left with $\ell_v(x) = x + t$. So our target function is

$$f(x) = \ell_v(g(\ell_h(x)) = m(x - s) + t.$$

This is the **point-slope form** of a line through $(s, t)$ with slope $m$. I will also say that this is **a line with slope $m$ centered at the point** $(s, t)$. Note that there are multiple ways to write the same line in point-slope form.

After lines, the next simplest kind of functions are the quadratic functions. Like lines, they also have a version of the point-slope form, which we will call the point-curvature form. Unlike point-slope form for lines, the point-curvature form of a quadratic equation is unique!

All quadratics have the same general shape, called a parabola:



They can be described with just two pieces of information: their vertex and their curvature. Given a quadratic function $ax^2 + bx + c$, its curvature is simply the coefficient $a$. If you know the vertex is at $(s, t)$, then you can reconstruct your parabola by translating $ax^2$ horizontally $s$ units and vertically $t$ units (since $ax^2$ has its vertex at $(0, 0)$).

Repeating the work in the previous section, we want to compose $ax^2$ with $x + t$ on the left and $x - s$ on the right to achieve this. This results in

$$f(x) = a(x - s)^2 + t$$

This is called the **point-curvature form of a parabola/quadratic equation passing through** $(s, t)$ **with curvature** $a$.

When the curvature $a$ is positive, the parabola opens upward, and when the curvature is negative it opens downwards. A larger $a$ value corresponds to a "pointer" parabola, while smaller $a$ values give rise to flatter parabolas.

Notice that if we expand the point-curvature form, we get

$$f(x) = ax^2 + \underbrace{-2sa}_{=b}\,x + \underbrace{as^2 + t}_{=c}$$

It follows that the vertex $(s, t)$ of the parabola $ax^2 + bx + c$ is at

$$s = \frac{b}{-2a}$$

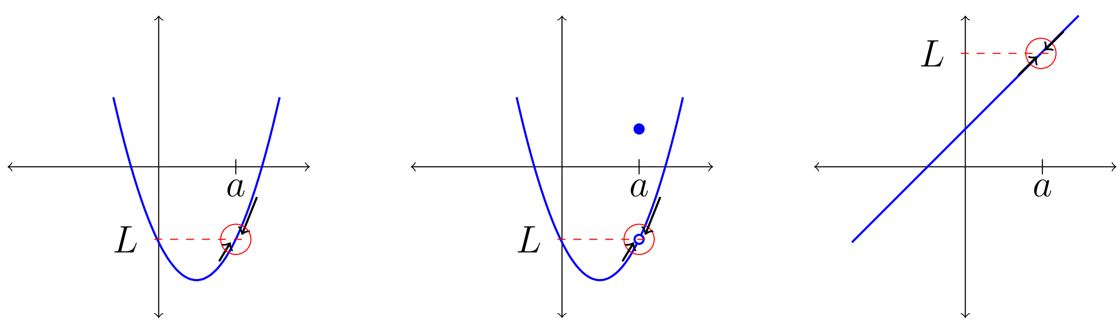
$$t = c - as^2$$

An important application of calculus is finding local approximations to functions. In other words, given a function $f(x)$ and point $x = a$, we want another function $g(x)$ which is very close to $f(x)$ near $a$. We will use limits to make this idea quantitative.

Let $f(x)$ be a function that we are interested in understanding near $x = a$ ($f(x)$ is not necessarily defined at $a$). We use the following notation to express the limit of $f(x)$ as $x$ heads to $a$:

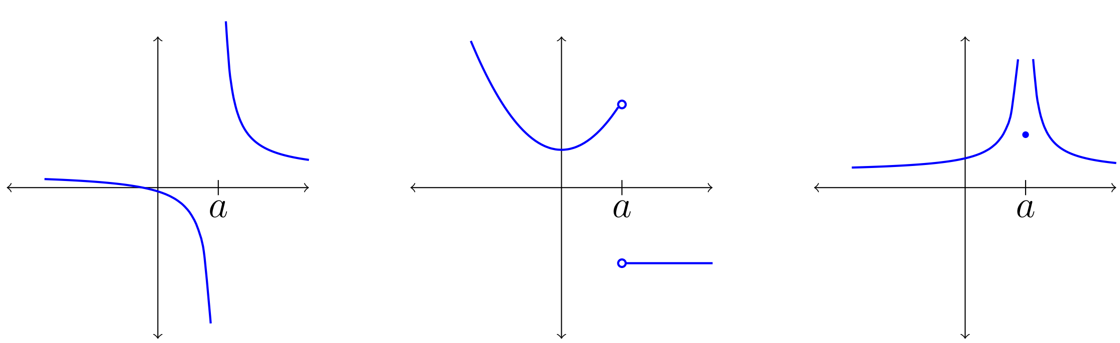
$$\lim_{x \to a} f(x) = L$$

This means that as $x$ gets closer and closer to $a$, the value of $f(x)$ must get closer and closer to $L$. Here are some examples with pictures. The limit point over $x = a$ is circled in red.




You can see that even though the second function is defined at $x = a$, the limit *does not* equal the value of the function at $x = a$. On the other hand, for the first and third functions the value of the limit is exactly the same as the value of the function. This is because those functions are **continuous**: if we trace along them towards $a$, there are no sudden jumps.

Sometimes, limits don't exist at all. This can happen in a few different ways:



In the first example, the function is headed down to $-\infty$ from the left, and $+\infty$ on the right. In order for the limit to exist, the limits on both sides have to exist also.

In the second example, the function does have a limit if you approach from only the right or left (corresponding to filling one of the open circles) but those two values are different, so the overall limit doesn't exist.

In the third example, the two sides are headed to the same place, $+\infty$, but that is not a number. However, when this happens we may sometimes write $\lim_{x\to a} f(x) = +\infty$ since this does still give us the useful information that the function takes arbitrarily large values as we move towards $x = a$.

Limits tell us what a function is doing near a point at which it may not be defined. The most common reason (for us) that a function will not be defined at a point is that it involves division. Consider the following rational functions:

$$\frac{x^2+1}{x-3} \qquad \frac{x^2-4}{x+2} \qquad \frac{x^3-x}{x}$$

They cannot be defined at $x = 3$, $x = -2$, and $x = 0$, respectively, because that would require division by zero. If you sketch graphs of these functions, you can see that the limit of the first does not exist at $x = 3$, but the other two seem to have limits (double check that in your sketch the first looks like a line with a hole, and the second like a parabola with a hole).

Here is how we can rigorously calculate those limits:

$$\begin{aligned}
\lim_{x\to a} f(x) &= \lim_{x\to -2} \frac{x^2-4}{x+2} \\
&= \lim_{x\to -2} \frac{(x+2)(x-2)}{x+2} \\
&= \lim_{x\to -2} x-2 && \text{when } x \neq -2 \text{ we can cancel the } x+2 \\
&= -2-2 = -4
\end{aligned}$$

Between the 2nd and 3rd lines, we use the important fact that the limit is about $x$ values heading *towards* $x = -2$, but which do not equal $x = -2$. That means $x + 2$ will never be zero, so it is perfectly safe to divide by it, which cancels with a factor in the numerator! After doing so, we end up with the line $x - 2$, which is continuous, so we can simply plug in $x = -2$ to finish evaluating the limit.

As mentioned at the start of today's notes, one goal of calculus is to find approximations to functions. We want to make the intuitive statement "$f(x)$ is very close to $g(x)$ near $x = a$" into a quantitative statement. Limits let us do this.

When functions approximate each other, we call their difference an error:

$$error(x) = f(x) - g(x)$$

To say that $f(x)$ and $g(x)$ are close near $x = a$ is the same as requiring that the error is small in a precise sense: for example, the most common scenario in calculus is when we want to approximate $f(x)$ at $x = a$ by a linear function $l(x) = m(x - a) + b$:

$$f(x) = m(x - a) + b + error(x)$$

If this is a very good approximation, then the error should be much smaller than the main linear term $m(x - a)$. One way to quantify the statement "$error(x)$ is less than linear at $x = a$" is as follows:

$$\lim_{x \to a} \frac{error(x)}{x - a} = 0$$

The numerator and denominator are both heading to zero as $x$ goes to $a$, so for this limit to exist and equal zero, the numerator must be going to zero much faster than $x - a$. To simplify our work, we will introduce a little notation:

$$\text{we write } \; f(x) \underset{x \approx a}{<<} g(x) \; \text{ when } \; \lim_{x \to a} \frac{f(x)}{g(x)} = 0$$

As indicated last time, our current goal is approximate functions by lines. Even complicated functions built out of square roots, polynomials, and other operations still look almost linear if you zoom in closely enough, and it's much easier to work with a line than any other kind of function.

Our goal is as follows: given a function $f(x)$, we want to find a linear function $\ell(x)$ which is the **best possible linear approximation** to $f(x)$ near $x = a$. In other words,

$$f(x) = \ell(x) + error(x)$$

where the error should be much smaller than linear at $x = a$. For convenience, we will put the line in point-slope form, centered at $a$:

$$f(x) = m(x - a) + b + error(x)$$

Because the error is smaller than linear at $x = a$, then it has to be zero at $x = a$, so we can determine the constant term of $\ell(x)$ in point-slope form by evaluating at $x = a$:

$$f(a) = m(a - a) + b + error(a) = b$$

This reduces our goal to finding just a slope

$$f(x) = m(x - a) + f(a) + error(x)$$

such that the error will be very small.

Using what we learned last time, we can make quantitative the notion that the error is much less than linear: we require that

$$error(x) \underset{x \approx a}{<<} x - a$$

which means that the error, as a function of the distance of $x$ from $a$ decreases faster than linearly as $x$ moves towards $a$. This is equivalent to the following limit being zero:

$$\lim_{x \to a} \frac{error(x)}{x - a} = 0$$

Let's look at some simple examples with $a = 0$.

**Example 1.** Let $f(x) = x^2 + 7x - 5$. There is one linear component clearly visible and in the correct point-slope form at $x = 0$: namely, $7x - 5$. Let's try this guess out:

$$f(x) = 7x - 5 + error(x)$$

Comparing the two sides, we see that $error(x) = f(x) - (7x - 5) = x^2$. Near zero, we expect that $x^2$ is much smaller than linear, because the square of a small number $x$ is much much smaller than the original $x$.

Indeed, we want to check that

$$x^2 \underset{x\approx 0}{<<} x$$

Which is the same as

$$0 \overset{?}{=} \lim_{x\to 0} \frac{x^2}{x} = \lim_{x\to 0} x \overset{\checkmark}{=} 0$$

In general, any higher powers $x^2, x^3, x^4, ...$ are much smaller than linear near $x = 0$. So it is typically straightforward to find the best linear approximation to a polynomial near $x = 0$ – just pull off the linear and constant terms. For example:

$$f(x) = x^7 - 3x^2 + x + 4$$

Then we know that $\ell(x) = x + 4$ and $error(x) = x^7 - 3x^2$ is all of the higher degree terms that are left over.

What if we are at $x = 1$ instead of $x = 0$? This is harder. Suppose we are given a polynomial in a nice form, though:

$$f(x) = (x-1)^3 + 2(x-1)^2 + 3(x-1) - 1$$

We can still identify a linear term $\ell(x) = 3(x-1) - 1$ in the correct form. The error this leaves is $(x-1)^3 + 2(x-1)^2$, which is $\underset{x\approx 1}{<<} x - 1$:

$$\lim_{x\to 1} \frac{(x-1)^3 + 2(x-1)^2}{x-1} = \lim_{x\to 1}(x-1)^2 + 2(x-1) \overset{\checkmark}{=} 0$$

This suggests that to find linear approximations of polynomials at points $x = a$ other than $x = 0$, we need to rewrite them in terms of powers of $x - a$ to make identifying the linear term straightforward. For now, there are three basic methods (and we will learn a much faster fourth one next week).

We will apply all of these methods to $f(x) = x^2 + 3x + 1$ at $x = 2$.

**Method 1. "Undetermined Coefficients".** Our goal is to rewrite $f(x)$ in the following form:

$$f(x) = A(x-2)^2 + B(x-2) + C$$

What we can do is expand this expression and compare to the original version of $f(x)$, treating $A, B, C$ as variables. We will get a system of linear equations in $A, B, C$ that we can solve.

$$\begin{aligned} f(x) &= A(x-2)^2 + B(x-2) + C \\ &= A(x^2 - 4x + 4) + Bx - 2B + C \\ &= Ax^2 - 4Ax + 4A + Bx - 2B + C \\ &= Ax^2 + (B - 4A)x + 4A - 2B + C \\ f(x) &= x^2 + 3x + 1 \end{aligned}$$

Compare the leading coefficients: we see that $A = 1$.

From the linear coefficients, we get

$$B - 4A = 3$$

Since $A = 1$, this is the same as $B - 4 = 3$, so $B = 7$.

Lastly, looking at the constant term, we have

$$C + 4A - 2B = 1$$

After plugging in $A = 1$ and $B = 7$, we have $C + 4 - 14 = 1$, hence $C = 11$.

This puts $f(x)$ in the desired form:

$$f(x) = (x-2)^2 + 7(x-2) + 11$$

So the best linear approximation at $x = 2$ is $\ell(x) = 7(x-2) + 11$, with an error of $(x-2)^2$ which is smaller than linear near $x = 2$.

**Method 2. "Trick Re-Expansion".** This method involves a small trick. Notice that we want to have $x-2$'s everywhere instead of just an $x$. What we can do is add and subtract 2 from every $x$ – in total this means you've added zero, so it doesn't change anything at all:

$$\begin{aligned} f(x) &= x^2+3x+1 \\ &= (x+0)^2+3(x+0)+1 \\ &= (x-2+2)^2+3(x-2+2)+1 \end{aligned}$$

Then we re-expand by distributing *while treating each of the $x-2$'s as a single unit.*

$$\begin{aligned} f(x) &= (x-2+2)^2+3(x-2+2)+1 \\ &= ([x-2]+2)([x-2]+2)+3([x-2]+2)+1 \\ &= (x-2)^2+4(x-2)+4+3(x-2)+6+1 \\ &= (x-2)^2+7(x-2)+11 \end{aligned}$$

This once again puts the function $f(x)$ in the desired form, where we can easily read off that the linear approximation is $\ell(x)=7(x-2)+11$ with error term $(x-2)^2$ much smaller than linear near $x=2$.

**Method 3. "Translation".** In the past, this has generally been the most popular method. It is a little more conceptual, but can be easier to compute in practice. We start as follow:

$$f(x)=m(x-2)+b+error(x)$$

As was pointed out earlier, we can evaluate this at 2, which immediately tells us

$$b=f(2)=2^2+3*2+1=4+6+1=11$$

So all that remains to do is find the slope. The key ingredient is that **if you translate the graph of $f(x)$, the best linear approximation is translated with it, and the slope doesn't change**. So what we will do is translate $f(x)$ left by 2. This gives us a new function whose best linear approximation's slope at $x = 0$ is the same as the slope we are looking for here at $x = 2$.

Let $g(x) = f(x+2)$ be the translation of $f(x)$ to the left by 2. The slope of $g(x)$ at $x = 0$ is the same as the slope of $f(x)$ at $x = 2$. To find the slope of $g(x)$ at $x = 0$, we want to expand it:

$$\begin{aligned} g(x) &= f(x+2) \\ &= (x+2)^2 + 3(x+2) + 1 \\ &= x^2 + 4x + 4 + 3x + 6 + 1 \\ &= x^2 + 7x + 11 \end{aligned}$$

The linear approximation of $g(x)$ at $x = 0$ can be read off immediately from this (it's $7x + 11$) and the slope is 7.

So if we return to

$$f(x) = m(x-2) + 11 + error(x)$$

we now know that $m = 7$. What is interesting about this method is that it does not tell us much about the error term. We know that

$$f(x) = 11 + 7(x-2) + error(x)$$

and that let's us write the error out,

$$\begin{aligned} error(x) &= x^2 + 3x + 1 - (11 + 7(x-2)) \\ &= x^2 + 3x + 1 - 11 - 7x + 14 \\ &= x^2 - 4x + 4 \end{aligned}$$

It is much less obvious that this error is much smaller than linear at $x = 2$!!

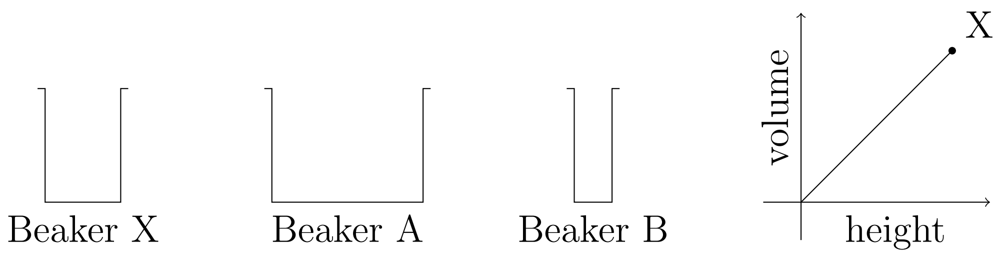
Let's double-check that $error(x) \underset{x \approx 2}{<<} x - 2$. It will require us to factor $error(x)$:

$$0 \stackrel{?}{=} \lim_{x \to 2} \frac{x^2 - 4x + 4}{x - 2} = \lim_{x \to 2} \frac{(x-2)^2}{x-2} = \lim_{x \to 2} x - 2 \stackrel{\checkmark}{=} 0$$

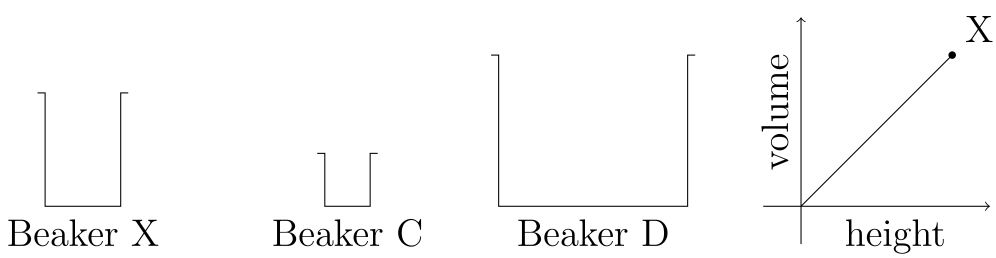
*Re-typed version of a worksheet from Yu-Wen Hsu.*

**1.** Describe how the width of the bottle at a point affects the shape of the calibration function. If the width is fixed, what shape is the function?

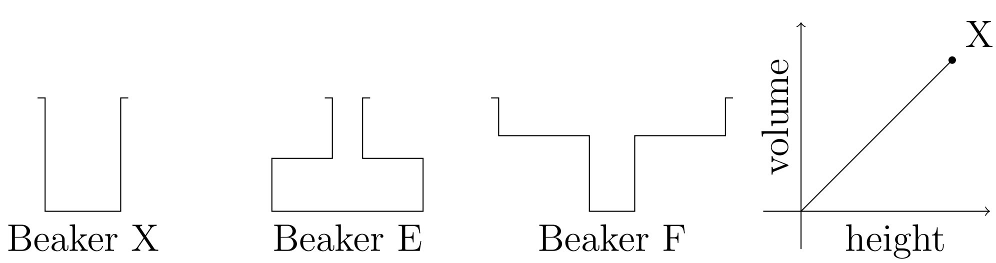
**2.** The graph below shows the calibration function for Beaker X. Sketch the calibration function for Beaker A and Beaker B.



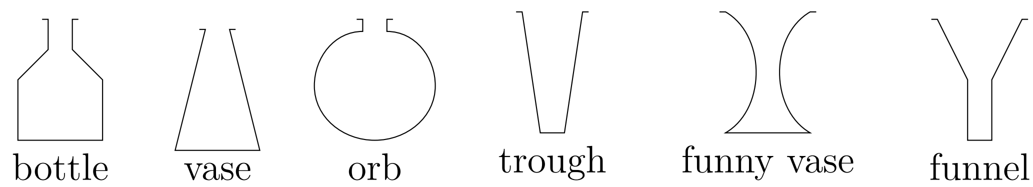

**3.** Repeat with Beaker C and Beaker D.

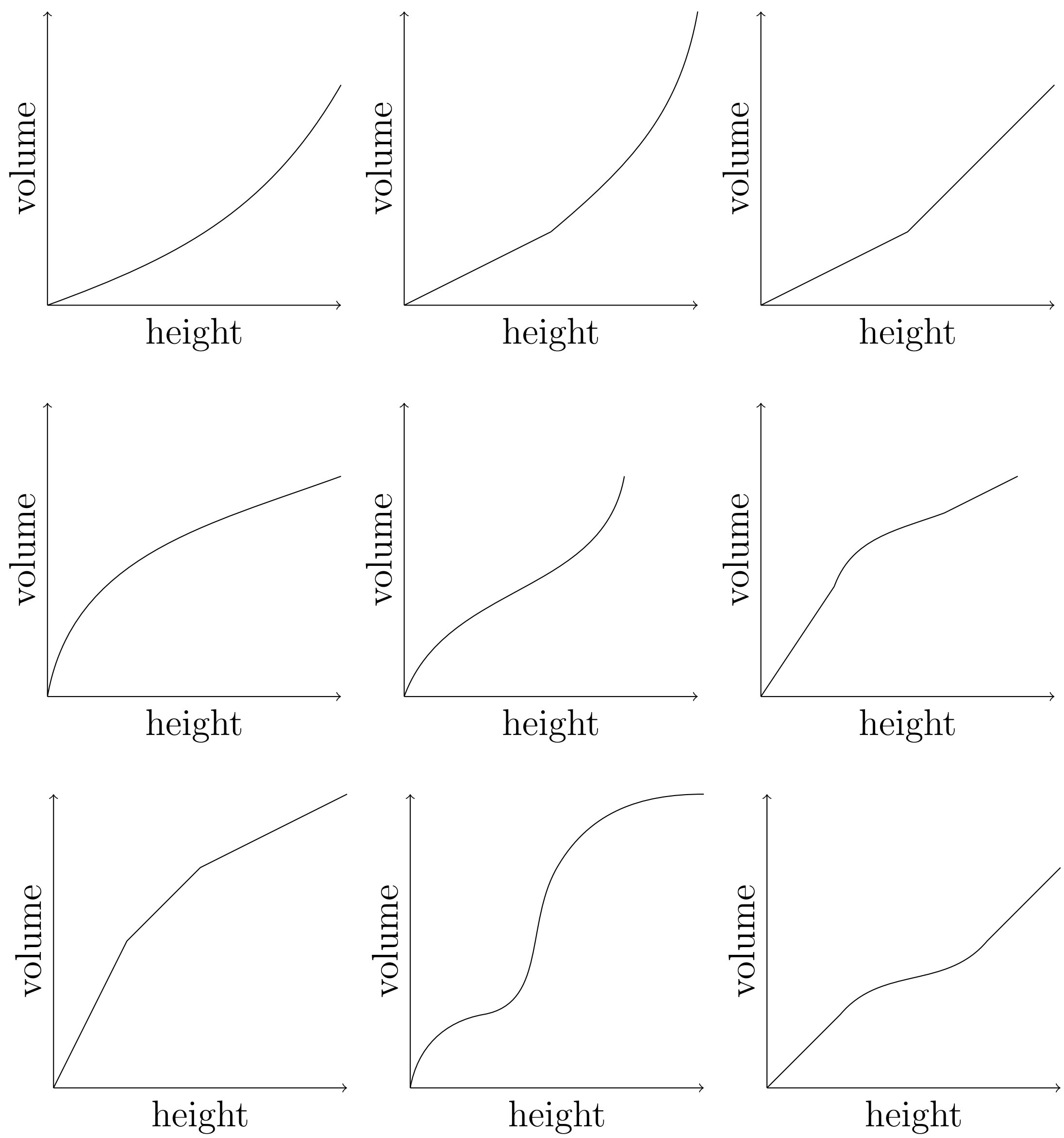



**4.** Repeat with Beaker E and Beaker F.



**5.** Six bottles are picture below. Identify the corresponding calibration graph for each bottle. For the three leftover bottles, sketch a calibration graph.

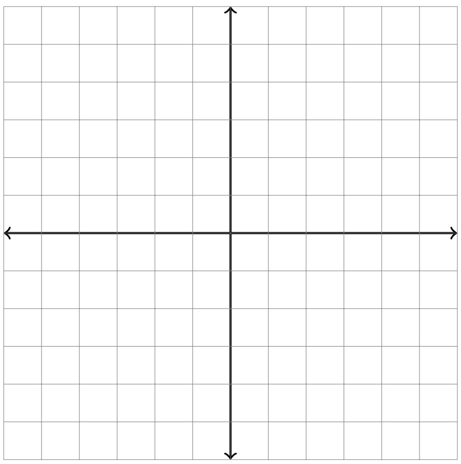


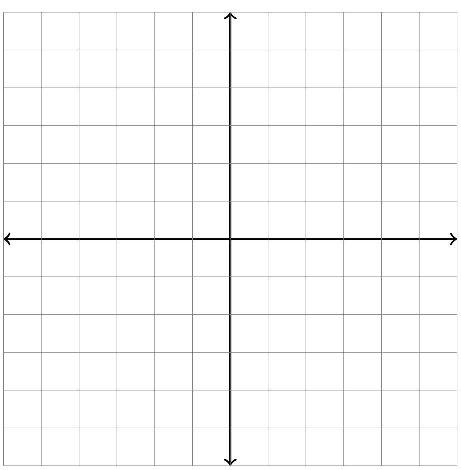
For each kind of function below, list a few example functions of that kind, and sketch some of their grapns. List some situations in "real life" which could be associated to such a graph. For example, a ball rolling towards a wall and bouncing off will give rise to an absolute value shaped graph if you plot distance from the wall against time.

Describe the domain and range of your functions.

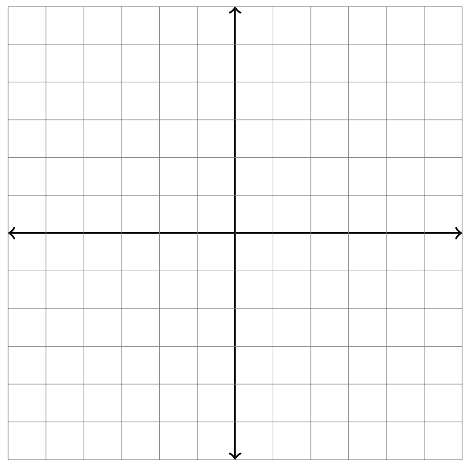
**(a) Linear**



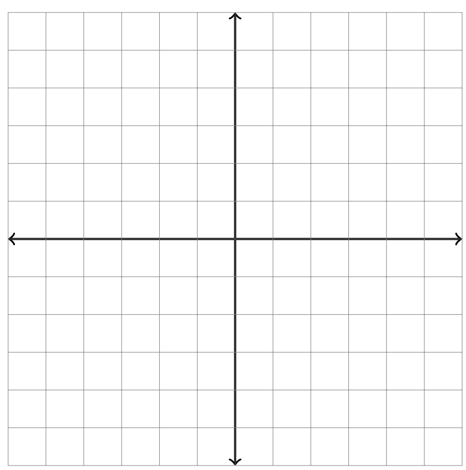
**(b) Quadratic**



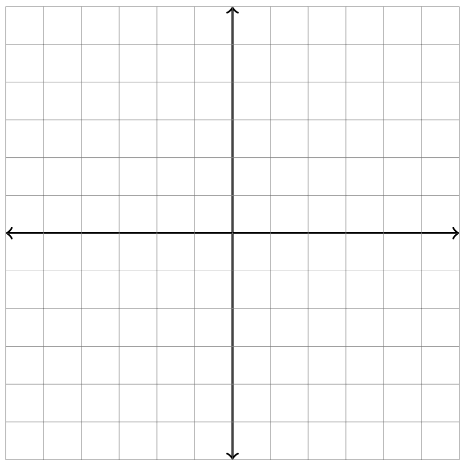
**(c) Square root**



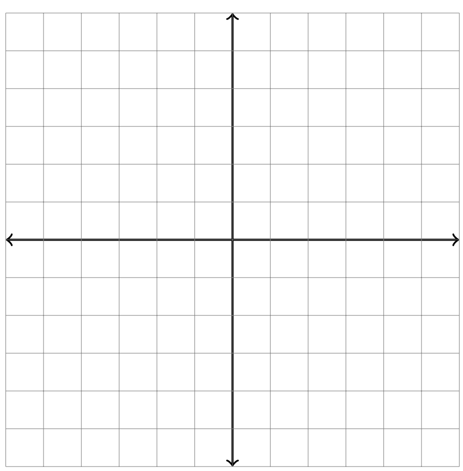
**(d) Polynomial**



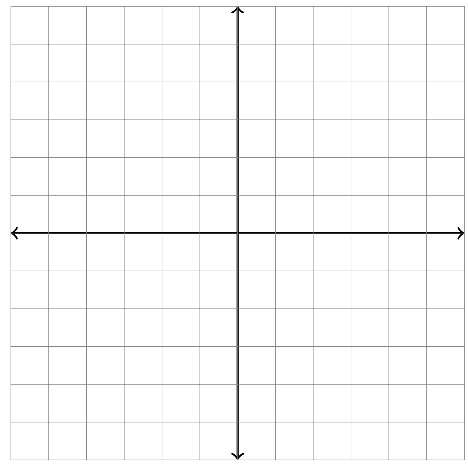
**(e) Absolute value**



**(f) Constant**

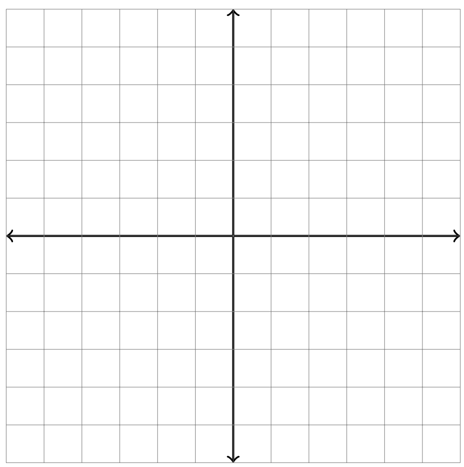


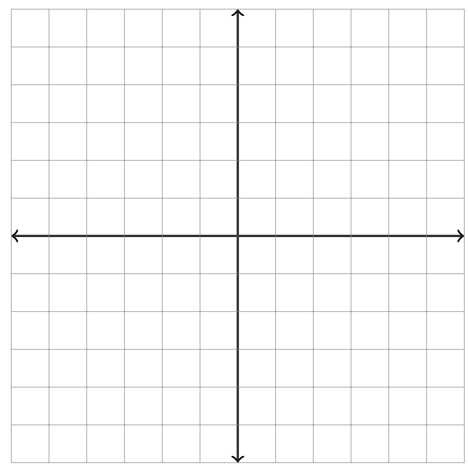
**(g) Piecewise**



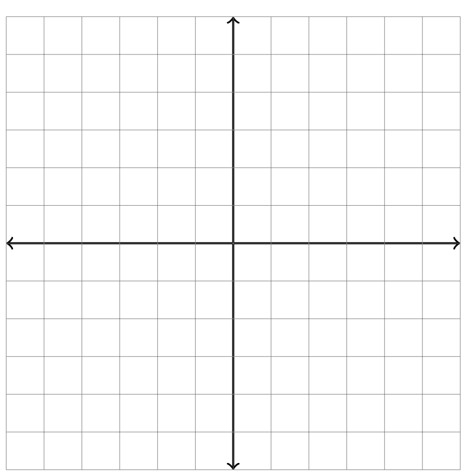
For each kind of the following problems, compute $f(g(x))$ and $f(g(x))$ and plot both.

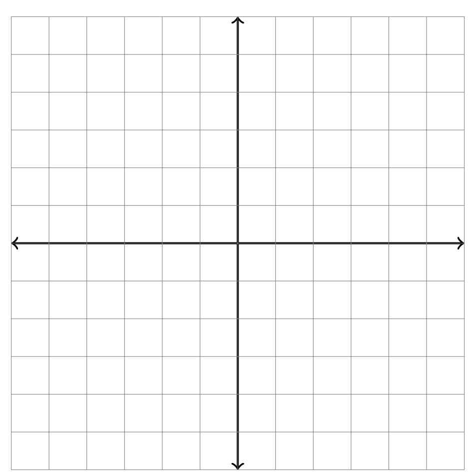
**(a)** $f(x) = x^2 + 1$, $g(x) = x^2 + x + 3$





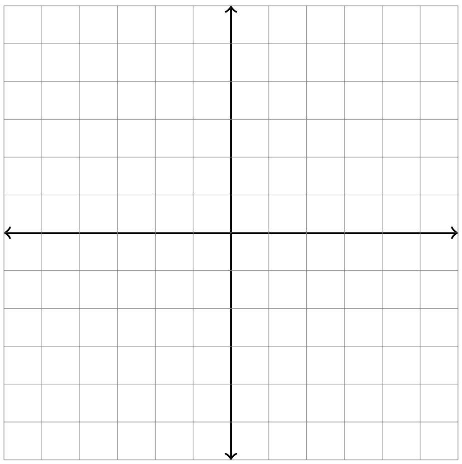
**(b)** $f(x) = \sqrt{x}$, $g(x) = x^2 + 2x + 1$

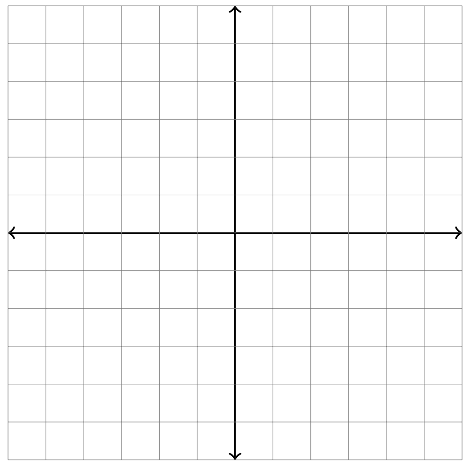




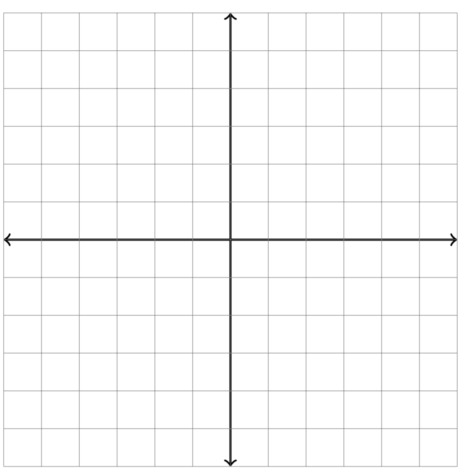
For each of the following, sketch $f(x)$ and then sketch $\ell(f(x))$ and $f(\ell(x))$.

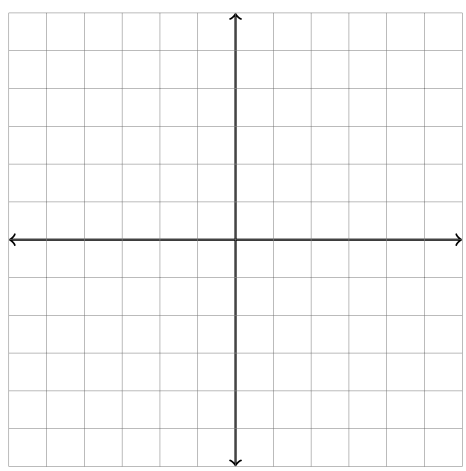
**(a)** $f(x) = x^3 - x$ and $\ell(x) = x - 3$



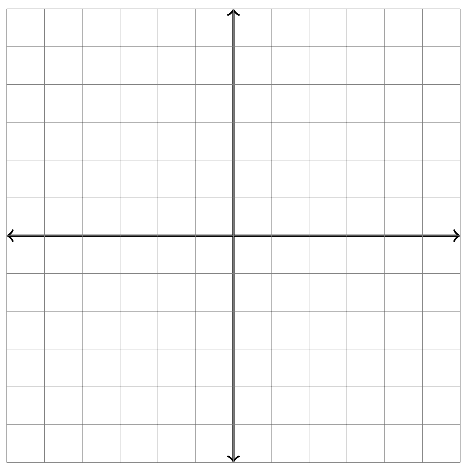


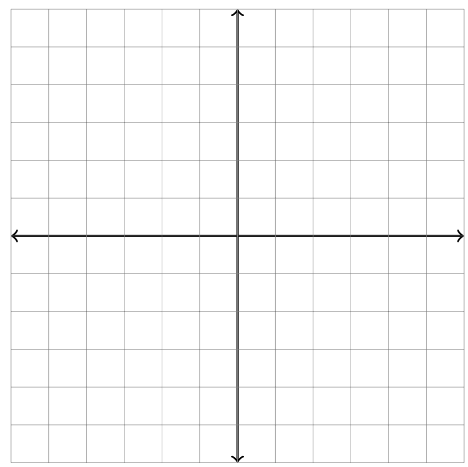
**(b)** $f(x) = x^2 - 2x - 1$ and $\ell(x) = 2x$



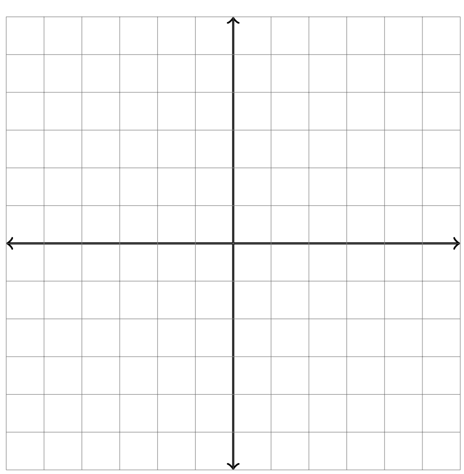


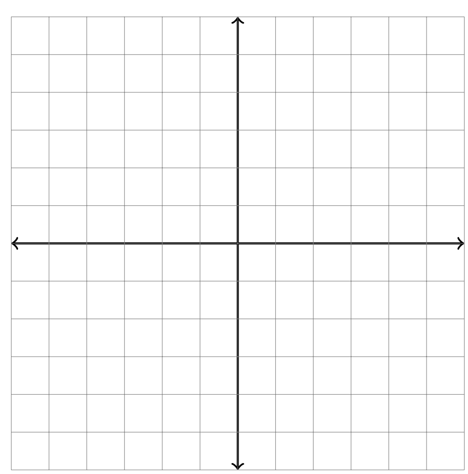
**(c)** $f(x) = |x + 2|$ and $\ell(x) = x/3$





**(d)** $f(x) = \sqrt{x^2 + 1}$ and $\ell(x) = x + 1$





**1.** State an example of each of the following kinds of function. Then, draw a sketch of each kind of functions. The sketch does not need to correspond to your example.

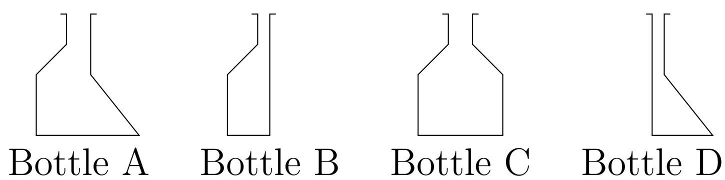
(a) linear

(b) quadratic

(c) absolute value

(d) polynomial

**2.** Sketch calibration functions for the following bottles. However, before making your sketches, describe how Bottles B and C are related, and also describe how Bottles A, B, and D are related. Explain how you can use this to simplify your sketching process.



**3.** Sketch $f(x)$ and $g(x)$ by making tables of values.

$$f(x) = x^3 - 3x + 1 \qquad g(x) = \left|x^2 - 2x - 1\right|$$

**4.** Let $\ell(x) = 2x + 1$. Describe how the graph of $\ell(f(x))$ can be obtained from the graph of $f(x)$ (hint: break $\ell(x)$ into two steps). Also describe how the graph of $f(\ell(x))$ is related to the graph of $f(x)$.

**5.** For the following pairs of linear functions, compute $\ell(x) + m(x)$, $\ell(m(x))$, and $m(\ell(x))$. Do you notice anything? (see Question 6)

(a) $\ell(x) = 2x + 1$ and $m(x) = -3x - 5$

(b) $\ell(x) = 10x + 22$ and $m(x) = 7 - x$

**6.** Let $\ell(x)$ and $m(x)$ be two linear functions. Explain how the slopes of $\ell(x) + m(x)$ and $\ell(m(x))$ can be obtained from the slopes of $\ell(x)$ and $m(x)$.

**1.** Consider the following linear functions:

$$a(x) = 4x - 3 \qquad b(x) = -x + 8 \qquad c(x) = 3x + 4$$

Determine the slopes of the following combinations of those functions without expanding them. Show your work.

(a) $a(x) + b(x)$

(b) $4c(x) + a(x)$

(c) $a(c(x)) - 5$

(d) $c(a(x) + 2b(x))$

**2.** Describe in words how the following linear functions transform a graph when applied on the left (in other words, compare the graphs of $\ell(f(x))$ and $f(x)$):

(a) $2x + 4$

(b) $-3x - 1$

(c) $x + 20$

(d) $\frac{1}{4}x + 1$

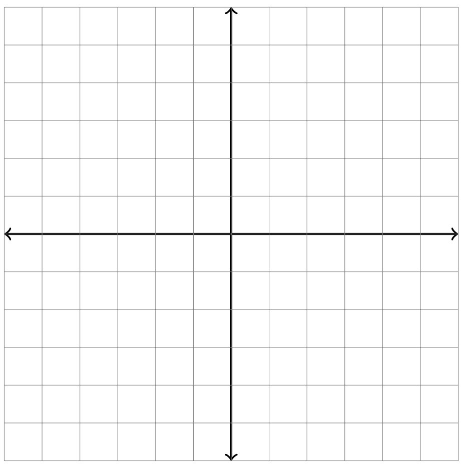
(e) $\frac{3}{2}x - 5$

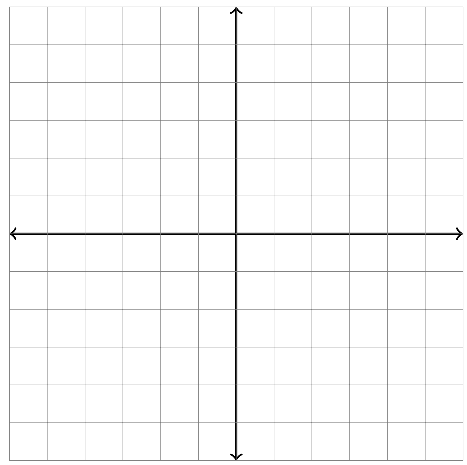
**3.** Using the same functions as in Question 2, but describe how the linear functions transform a graph when applied on the right (i.e. $f(\ell(x))$).

**4.** Using the functions $a(x), b(x), c(x)$ from Question 1, calculate all of the products $a(x)b(x), a(x)c(x), b(x)c(x)$. What is the linear coefficient of each of the resulting functions? What do you notice?

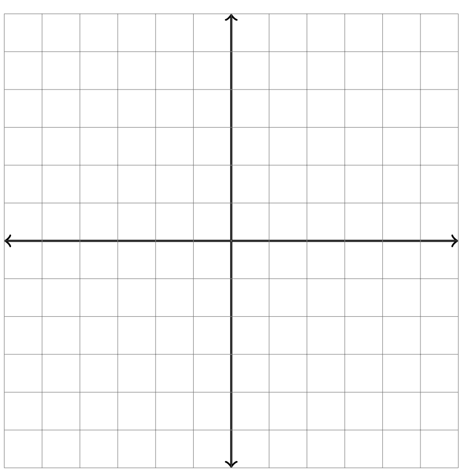
For each kind of the following problems, graph both $f(x)$ and $g(x)$, then determine a linear map which transforms one into the other.

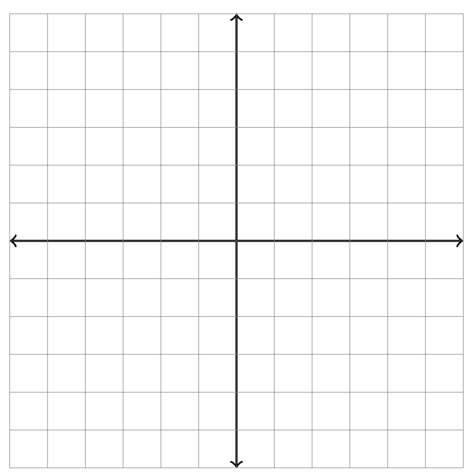
**(a)** $f(x) = x^2 + 1$, $g(x) = x^2 - 4x + 2$



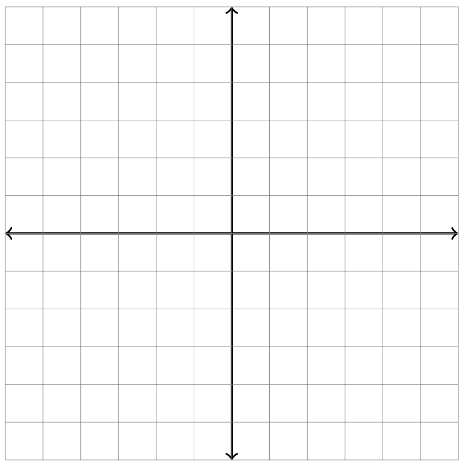


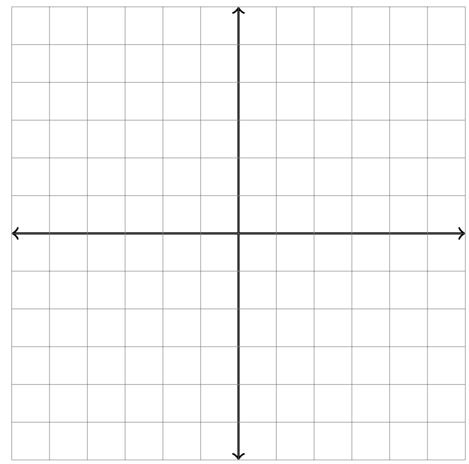
**(b)** $f(x) = 4x^2 + 2$, $g(x) = x^2 + 2$



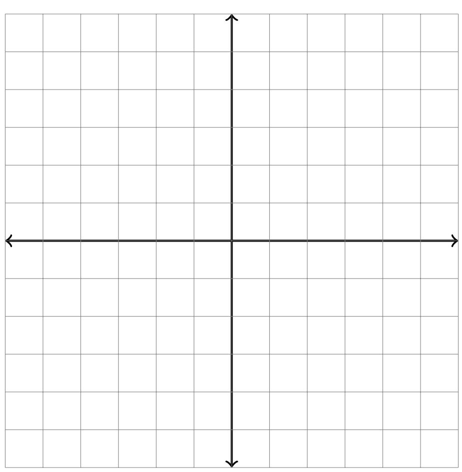


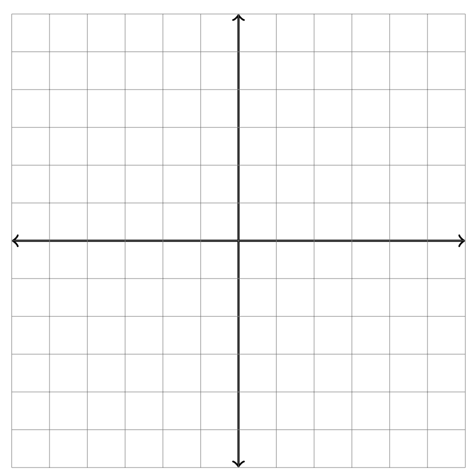
**(c)** $f(x) = x^2 + 2x - 1$, $g(x) = x^2 + 2x + 1$





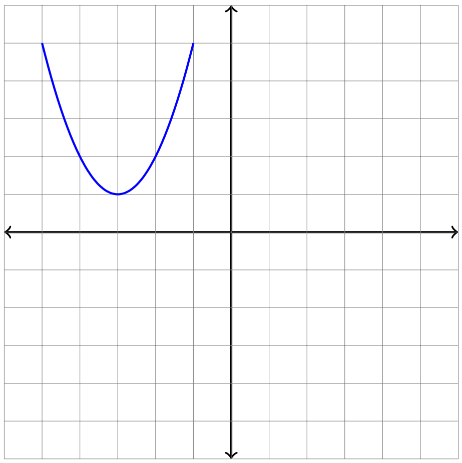
**(d)** $f(x) = 2x^3 - 4x$, $g(x) = x^3 - 2x$

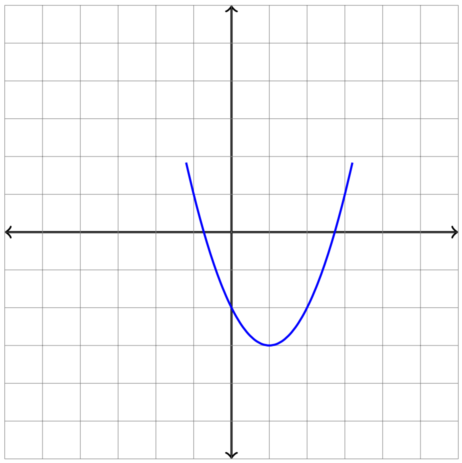




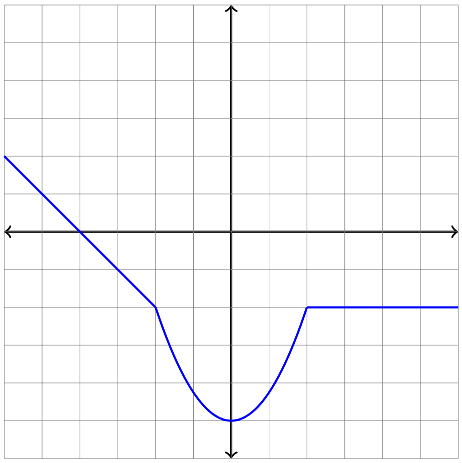
For each of the following pairs of graphs, write down linear functions which transform one into the other:

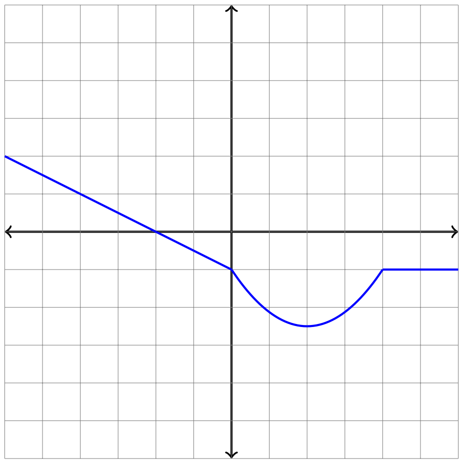
**(a)**





**(b)**





For each of the linear transformations, describe how it will change the graph of $f(x) = x^3 + 2x - 1$ when you look at $\ell(f(x))$ and $f(\ell(x))$. Then, expand those compositions.

**(a)** $\ell(x) = -x - 1$

**(b)** $\ell(x) = 2 + 4x$

We saw on the previous homework that combining bottles horizontally corresponds to adding their calibration functions. Suppose you combined bottles vertically: describe how this combines their calibration functions.

Practice factoring the following polynomial expressions:

**(a)** $x^2 - 4x + 3$

**(b)** $9x^2 + 3x - 2$

Practice expanding the following products:

**(c)** $(x - 1)^2(x - 2)$

**(d)** $(3x + 1)(x + 4)(x - 3)$

**(e)** $(x-2)^3$

**(f)** $(x+a)^4$

Let $f(x) = 2x^2-2x+3$. Expand $f(\ell(x))$ for each of the following functions $\ell(x)$ and state the linear coefficient of the resulting polynomial in each case:
**(a)** $\ell(x) = x-2$

**(b)** $\ell(x) = x+1$

**(c)** $\ell(x) = x-8$

**(c)** $\ell(x) = x+3$

Let $f(x) = 2x^2 - 2x + 3$. Expand $f(x+a)$ and state the linear coefficient of this polynomial in terms of the constant $a$.

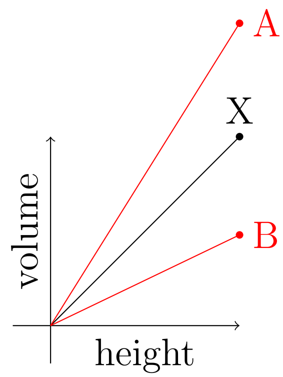
*Re-typed version of a worksheet from Yu-Wen Hsu.*

**1.** Describe how the width of the bottle at a point affects the shape of the calibration function. If the width is fixed, what shape is the function?

The width of the bottle tells you how steep the calibration function should be near the point. If the calibration function is a curve, a correspondingly steep (or shallow) line if we could zoom in very closely.

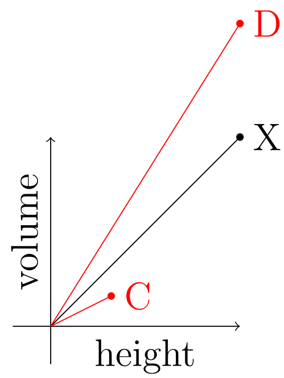
**2.** The graph below shows the calibration function for Beaker X. Sketch the calibration function for Beaker A and Beaker B.

The beakers are the same height, so the graphs go the same distance to the right, but end at very different volumes because Beaker A is much bigger and Beaker B is much smaller.



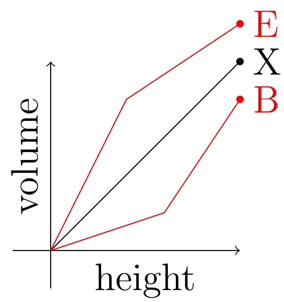

**3.** Repeat with Beaker C and Beaker D.

Here, Beaker C is much smaller than the other two, so we know its calibration function has to end sooner. It is narrower than Beaker X, so it will be shallower too. Beaker D should look about the same as Beaker A from the previous question.

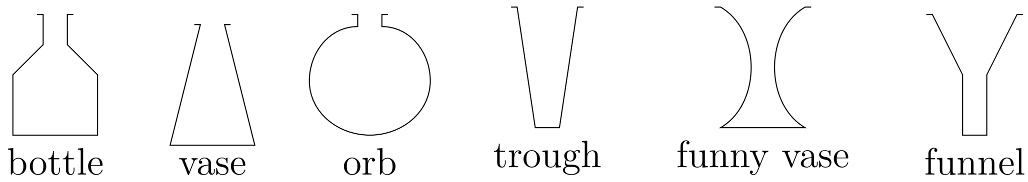



**4.** Repeat with Beaker E and Beaker F.

This is a lot harder! My approach is the view each beaker as two beakers of constant width stacked on top of each other. Beaker E begins wide, hence steep, then turns narrow, so it becomes shallower: we will connect a steep line to a shallow line. Beaker F does the opposite.



**5.** Six bottles are picture below. Identify the corresponding calibration graph for each bottle.




Bottle: starts at a constant width, then narrows to a constant width, so it should be a straight line followed by a curve that flattens into a straight line.

Vase: constant narrowing, so it should be a curve that starts to flatten.

Orb: width increases then decreases, so it should be a line curving up then another curve that flattens.

Trough: opposite the vase, it should curve upward.

Funny Vase: opposite the orbit, it starts wider, narrows, then widens, and so it should start as a flattening curve and then an increasing curve.

Funnel: starts as a constant width, hence straight line, and then widens, hence an upward curve.