

Now we are going to analyze the structure of polynomials a little more. A polynomial is any function that is built out of constants and the variable out of multiplication and addition.

Every polynomial can be “expanded” into the form

$$a_0 + a_1x + a_2x^2 + \dots + a_nx^n, \quad a_n \neq 0$$

for some nonnegative integer  $n$ .

Some important quantities:

- the integer  $n$  is called the **degree**
- the number  $a_n$  is the **leading coefficient**
- each “ $a_kx^k$ ” is called a **term**
- each term  $a_kx^k$  has two pieces: a **coefficient**  $a_k$  and a **monomial**  $x^k$ .

We’ve looked at lots of polynomials so far, here are a couple examples:

$$x + 1, \quad x^2 + 4x + 5, \quad x^{63} - x^{32} + 900$$

Most of the time, we think of high-degree polynomials as being “very big”, because a high power of a large number is enormous. But in this course, we are actually going to take sort of the opposite view: a high-degree term is **very small** if we evaluate it at a number less than 1.

For instance:

$$(0.5)^5 = \left(\frac{1}{2}\right)^5 = \frac{1}{32} \approx 0.03$$

and as we take higher and higher powers, the number gets smaller and smaller, decreasing by a factor of 2 at every step.

This might sound weird or uninteresting, but it is actually something we see all the time!! Let’s look at a decimal number like 3.1415. We can break it down into a sum of each of its pieces:

$$3.1415 = 3 + 0.1 + 0.04 + 0.001 + 0.0005$$

Secretly, this is a polynomial! Factor out the digit at each step:

$$= 3 + 1(0.1) + 4(0.01) + 1(0.001) + 5(0.0001)$$

and then remember that  $0.1^2 = 0.01$  and  $0.1^3 = 0.001$  and so on:

$$= 3 + 1(0.1) + 4(0.1)^2 + 1(0.1)^3 + 5(0.1)^4$$

So what we have is, underneath all the numbers, the same thing as taking the polynomial

$$3 + x + 4x^2 + x^3 + 5x^4$$

and evaluating it at the small number (0.1)!! From this perspective we see why the higher degree terms could be thought of as “small”.

You might recognize that 3.1415 is an approximation to  $\pi$ , but  $\pi$  goes on for many many more digits (infinitely many!). This is still sensible, and it suggests that “infinite degree” polynomials might be interesting, although only if we plug *small* numbers into them.

Continuing with  $\pi$ , there is some infinite polynomial  $P(x)$  that looks like

$$P(x) = 3 + x + 4x^2 + x^3 + 5x^4 + 6x^5 + \dots$$

where the “dot dot dot” means we keep going (since it’s not possible to fit infinitely many things on a page, we have to use this). For this special polynomial,  $P(0.1) = \pi$  exactly.

Notice that if we cut off this infinite polynomial after some point, we get a regular polynomial, and one that’s fairly close to the original polynomial (because higher degree terms are “small”). If we truncate at degree  $n$ , we call it an  **$n$ -th order polynomial approximation to  $P(x)$** .

This approximation could be used to approximate  $\pi$ , but if we are just interested in  $P(x)$  as a function, it also gives us good approximations (again sticking with small values of  $x$ ).

The main goal of calculus is to tell us how to find these polynomial approximations for LOTS of other functions besides infinite polynomials. Sticking these approximations together, we get a power series approximation to our function. These are super useful for practical applications, because computers and calculators can handle polynomials well, but don’t really have a way to talk about things like square roots or trig functions. Deep down, calculators use these approximations!

A very important equation in chemistry is the ideal gas law:

$$PV = nRT$$

For a container of some gas, it relates the pressure  $P$ , the volume of the container  $V$ , the amount of gas  $n$ , and the temperature  $T$ . The number  $R$  is a constant number (the ideal gas constant) which can be determined experimentally.

Since this is only for “ideal” gases, we are led to wonder about a non-ideal gas law. A very blunt way to do this is to introduce a new number  $z$ , and require that

$$PV = nRTz$$

Being non-ideal has to do with interactions between the gas molecules that aren't just collision. For instance, some gases can be “sticky” with each other (water vapor, for instance).

The number  $z$  is just a fudge-factor that should depend on how “non-ideal” the gas is, plus also the ambient pressure and temperature conditions. Volume isn't relevant if we agree that non-ideal behavior comes from weird collision behavior, since the way two molecules collides has nothing to do with the size of the container.

Suppose we also hold temperature constant and want to explore  $z$  as a function just of pressure. What if we think of  $Z$  as a power series?

$$Z = a_0 + a_1P + a_2P^2 + a_3P^3 + \dots$$

When  $P$  is very close to zero, collisions are extremely unlikely, and so even a non-ideal gas should probably behave ideally. This means we should get the regular ideal gas law if we set  $P = 0$ , which tells us  $a_0 = 1$ .

If  $z = 1$  then we just get the usual ideal gas law  $PV = nRT$ .

All of the higher-degree terms should measure some sort of defect in the gas's behavior, as far as collisions are concerned.

This turns out to be very physically meaningful:  $P^k$  is proportional to the probability of  $k$  gas molecules colliding at once, and so  $a_k$  should quantify the “bad behavior” this causes for the ideal gas law.

For gases at typical pressures, collisions of two molecules aren't unexpected, but they get less and less likely as we ask that more and more molecules collide, because molecules are so small and gases are relatively sparse.

So a truncation of the our power series should still enable us to get a quite accurate value for  $Z$ . This is indeed the case, and it recovers the “virial equation of state”. The coefficients can be determined experimentally by using calculus!!