We spent some time reviewing the material of the previous Thursday (see those notes!!). The main new topic for today is approximation.

The idea is that starting from

$$f(x) = f(0) + f'(0)x + error,$$

you can assume that for values of x very near zero, where the error is *extremely* small, the linear piece should give us quite accurate approximations; to compute such values, we'd only need $f(0)$ and $f'(0)$.

For instance, suppose there is a mystery function $f(x)$ where all we know is that $f(0) = 3$ and $f'(0) = -1$, and we are asked to find $f(1)$. This isn't possible without more information on $f(x)$, but we can at least make a plausible guess from the information we have:

$$\begin{aligned} f(x) &= f(0) + f'(0)x + error(x) \\ &= 3 - x + error(x) \\ &\approx 3 - x + \cancel{error(x)} \\ &= 3 - x \end{aligned}$$

leading us to guess $f(1) \approx 3 - 1 = 2$. We could guess other values using the same approximation. From the definition of the derivative, we expect the best results near zero.

The idea of approximation works just as well at other points, as long as we remember to use the adjusted definition of the derivative:

$$f(x) = f(a) + f'(a)(x - a) + error$$

which "simplifies", by dropping the error, to a linear function:

$$f(x) \approx f(a) + f'(a)(x - a)$$

that we expect to be more accurate the nearer x is to a.

It sometimes happens that we have some particular value of the function and several derivatives. By performing multiple approximations, we can sketch points on the graph.

For instance, suppose a pendulum is swinging and we have the following information about the height $h(t)$ as a function of time:

$$h(0) = 6, h'(0) = -3, h'(1) = -2, h'(2) = 2, h'(3) = 3$$

Then we can perform a sequence of approximations from 0 to 1, then 1 to 2 then 2 to 3 then 3 to 4.

$$h(1) \approx h(0) + h'(0)(1 - 0) = 5 - 3 = 3$$

$$h(2) \approx h(1) + h'(1)(2 - 1) = 3 - 2 = 1$$

$$h(3) \approx h(2) + h'(2)(3 - 2) = 1 + 2 = 3$$

$$h(3) \approx h(3) + h'(3)(4 - 3) = 3 + 3 = 6$$

Plotting this trajectory, we see that the pendulum's height follows a curved path, and reaches its minimum height at or near $t = 2$.