Today's topic is our most conceptually advanced, but I think it really displays the power and utility of calculus.

Previously, we defined e^x and other exponential functions as solutions to certain rate equations. The shape of then notation e^x certainly suggests that we are taking a number e and raising it to the power of x, but just from our definitions, we actually don't know this!! (even though it's true)

Our definition was motivated by problems of unrestricted growth, which we might not guess ahead of time have anything to do with exponentiation – it is remarkable and interesting that they really are the same phenomenon.

What we did today is bridge the gap between these two ideas: showing that the solutions to the rate equation behave exactly like exponentiation:

$$(e^x)^r = e^{rx} \quad \text{and} \quad e^{x+a} = e^x e^a.$$

Since you've probably seen these rules before and have already internalized them as properties of e^x, I want to emphasize again: ***these really are properties of e^x, but they just are not immediately visible from our definition!***.

We do know actual exponentiation behaves that way, for instance

$$(2^2)^3 = 2^{2*3} = 2^6 \quad \text{and} \quad 2^2 2^3 = 2^{2+3} = 2^5$$

It would have been a bad idea to write e^x in that way if these properties didn't hold.

Since we defined e^x by calculus, our best shot at checking those properties is with calculus. To keep things clear, let's just focus on the first: we want to somehow show that the two functions $f(x) = (e^x)^r$ and $g(x) = e^{rx}$ are actually equal.

The ***ESSENTIAL*** calculus tool for comparing functions is the Fundamental Theorem of the Derivative. Remember that this says that if we have a function $f(x)$ and we know that $f'(a) = 0$ for all a, then $f(x)$ must be a constant. The reason it is relevant is that we can take our goal of $f(x) = g(x)$ and rearrange it in two ways:

$$f(x) - g(x) = 0 \quad \text{or} \quad \frac{f(x)}{g(x)} = 1$$

In both cases, we have a new combined function that we hope is equal to a particular constant. The Fundamental Theorem of the Derivative is a tool for showing that functions are constant, so it's very relevant here. It turns out that in this situation, the fraction will be our best shot.

So our goal is to show $(e^x)^r = e^{rx}$. We have now rearranged this to

$$\frac{(e^x)^r}{e^{rx}} \underset{(want)}{=} 1$$

First, we will use the Fundamental Theorem of the Derivative to check that the left hand function is actually a constant, and then we will do a tiny bit of work to check that that constant is simply 1.

So let's differentiate the left hand side: first we use quotient rule

$$\frac{f(x)}{g(x)} \underset{derivative}{\longrightarrow} \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

We know $f(x)$ and $g(x)$ but we still need to figure out $f'(x)$ and $g'(x)$. Let's start with $f'(x)$.

The function $f(x)$ is $(e^x)^r$. This is a composition of two function: $H(x) = e^x$ on the inside and $K(x) = x^r$ on the outside. This means we can use the composition rule – the derivative of $f(x) = K(H(x))$ is $K'(H(x))H'(x)$. By power rule, $K'(x) = rx^{r-1}$, and by definition $H'(x) = e^x$. So what we get is

$$f'(x) = K'(H(x))H'(x) = r(e^x)^{r-1}e^x = r(e^x)^r$$

By very similar reasoning (check this yourself!!!) we can determine that $g'(x) = re^{rx}$. (it is a composition of e^x and rx - use composition rule)

That was a lot of calculation. Let's recap where we are:

$$f(x) = (e^x)^r \qquad f'(x) = r(e^x)^r$$
$$g(x) = e^{rx} \qquad g'(x) = re^{rx}$$

and our present goal is to check that the derivative of $\frac{f(x)}{g(x)}$ is zero. Now we have all the ingredients to plug into the quotient rule:

$$\frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} = \frac{r(e^x)^r e^{rx} - (e^x)^r re^{rx}}{(e^{rx})^2}$$

The two quantities on top are the same (rearrange the multiplication a little) and so when subtracted we just get zero!!!!!

So we really have verified that the derivative of $\frac{f(x)}{g(x)}$ is zero. ***So the fundamental theorem of the derivative tells us that*** $\frac{f(x)}{g(x)}$ ***is a constant.***

To figure out the constant, we can plug in *anything* for x. The best value to plug into an exponential function is zero, because it's built into the definition.

$$\frac{f(0)}{g(0)} = \frac{(e^0)^r}{e^{r0}} = \frac{1^r}{e^0} = \frac{1}{1} = 1$$

After all of that work (it was not easy!!) we know know that $\frac{f(x)}{g(x)}$ is a constant function equal to 1, and so $f(x)$ and $g(x)$ are exactly equal.

I want to recall now what we did in class, which was slightly different. In class, we showed that if we take any function $f(x)$ which satisfies the rate equation

$$f'(x) = rf(x)$$

$$f(0) = C$$

(for some constants r and C)

then that function is unique. This is true more generally: for "nice" rate equations, the solution is *unique*. If you continue in math or any field that uses a lot of math, you will see this seemingly-simple fact used again and again. It often happens that we don't fully understand functions we encounter in "real life" but we know or can measure how they grow. Calculus lets us use that growth equation to identify the function and help us study it.

[I changed the explanation from what we did in class based on your feedback – I hope this alternative view is more digestible or gives you more perspective on it. BUT I still want you to remember generally this special property of rate equations]

All we have left to check is one more identity: $e^x e^a = e^{x+a}$, where a is some constant. This runs parallel to the previous identity - some of the algebra is actually a little easier in this case.

As before, it turns out that division will be most successful here. For convenience, we'll give the functions short names:

$$f(x) = e^x e^a \qquad g(x) = e^{x+a}$$

And we have a three-step process to get to really check that $f(x) = g(x)$:

(1) Differentiate $\frac{f(x)}{g(x)}$ with the quotient rule. This requires calculating $f'(x)$ and $g'(x)$. The second one needs composition rule.

(2) The above should be zero. ***By the fundamental theorem of the derivative, we then know that*** $f(x)/g(x)$ ***is a constant.***

(3) Plug in a convenient value for x to determine what the constant is – it should be 1, which will mean the two functions are equal.

(1) $f(x)$ is e^x times the constant e^a, so its derivative is just the derivative of e^x (which is e^x) times that same constant. In other words, we just end up with $f'(x) = f(x)$.

The same happens with $g(x)$ for slightly different reasons. This function is the composition of $H(x) = e^x$ with $K(x) = x + a$. The derivative of $K(x)$ is 1 (the constant 1) and the derivative of $H(x)$ is just $H(x)$. So the composition rule works out nicely:

$$H'(K(x))K'(x) = H'(K(x)) = H(K(x)) = g(x).$$

For both functions, it turns out that they equal their derivative, $f'(x) = f(x)$ and $g'(x) = g(x)$. As remarked above, this means the two will satisfy the same rate equation as long as their "starting populations" match, and that will for the whole entire functions to be equal.

(2) But to proceed in the same fashion as we did earlier in these notes, we will use the quotient rule and check that it's zero:

$$\frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} = \frac{f(x)g(x) - f(x)g(x)}{g(x)^2} = 0$$

(the top terms end up cancelling)

Thus, **by the fundamental theorem of the derivagive**, $\frac{f(x)}{g(x)} = \frac{e^x e^a}{e^{x+a}}$ is a constant.

(3) To find the constant, we can plug in anything we want for x. Like before, with exponential functions we are best able to handle $x = 0$. And things work out nicely:

$$\frac{e^0 e^a}{e^{0+a}} = \frac{e^a}{e^a} = 1$$

So the constant function $f(x)/g(x)$ is equal to 1. Rearranging $\frac{f(x)}{g(x)} = 1$, we get $f(x) = g(x)$. That's exactly what we wanted: $e^x e^a = e^{x+a}$!!!!